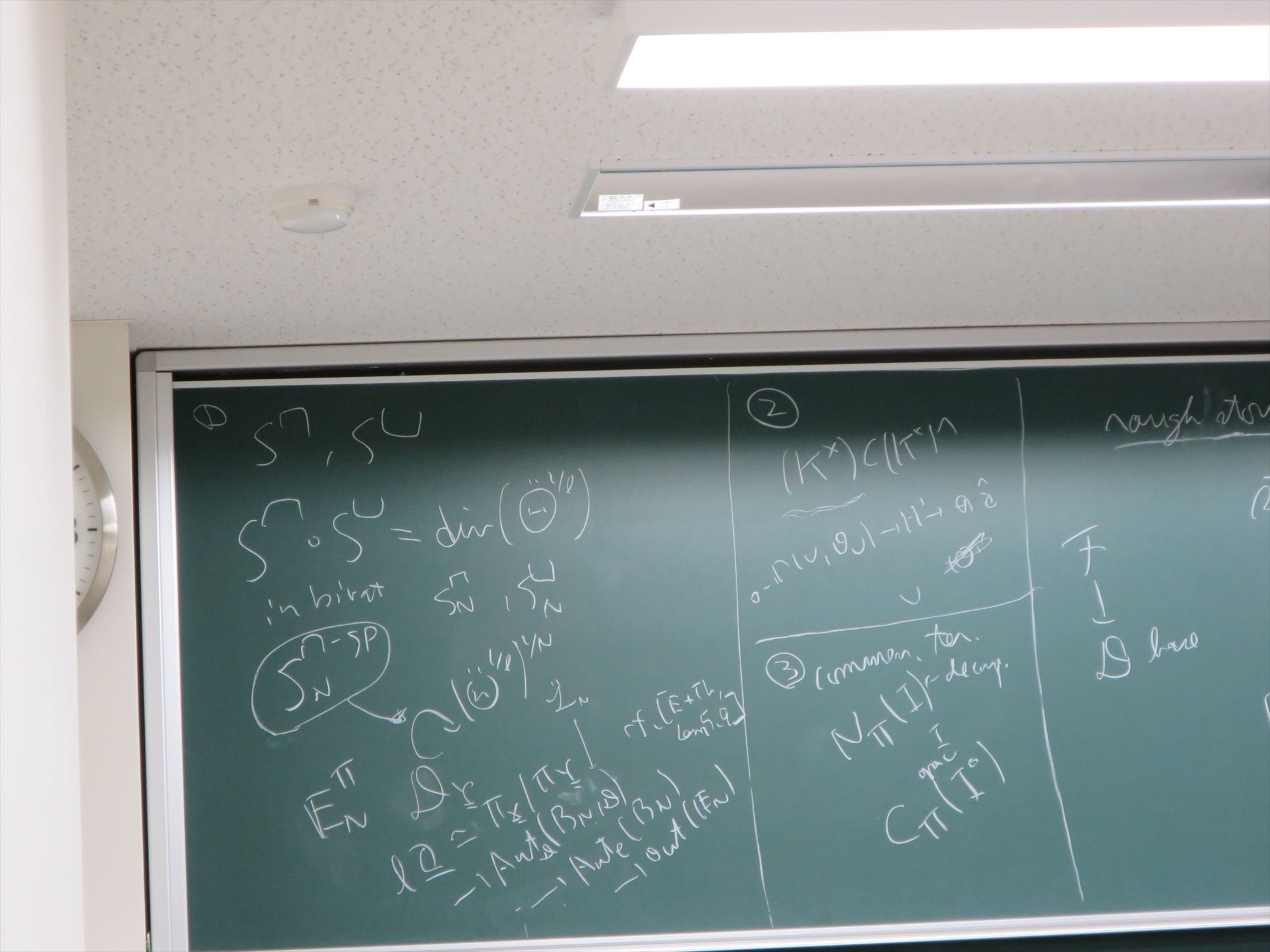


① $\mathfrak{S}^{\sqcap}, \mathfrak{S}^{\sqcup}$

$\mathfrak{S}^{\sqcap} \circ \mathfrak{S}^{\sqcup} = \mathrm{dir}\left(\ddot{\circledcirc}^{\,\pm/\ell}\right)$

in birat $\mathfrak{S}^{\sqcap}_N, \mathfrak{S}^{\sqcup}_N$

$\mathfrak{S}^{\sqcap\text{-}\mathrm{sp}}_N \curvearrowright \left(\ddot{\circledcirc}^{1/\ell}\right)^{1/N}$ $\mathfrak{I}_N$

cf. [EtTh, Lem 5.9]

$\mathbb{E}^{\Pi}_N$ $\mathcal{D}_{\underline{Y}}$

$\ell\underline{\Omega} \simeq \Pi_{\underline{X}}/\Pi_{\underline{Y}}$

$\to \mathrm{Aut}_{\mathcal{D}}(\mathbb{B}_N, \mathcal{D})$

$\to \mathrm{Aut}_e(\mathbb{B}_N)$

$\to \mathrm{Out}(\mathbb{E}_N)$

② $(K^{\times}) \subset ((\Gamma^{\times})^{\wedge}$

$0 \to \Gamma(V, \mathcal{O}_V) \to \Gamma' \to \Theta\hat{c}$

~~Θ~~

V

③ commen. ter.

$N_{\Pi}(I)$ r-decomp.

I

gpac

$C_{\Pi}(I^{\circ})$

rough

$\mathcal{F}$

$\downarrow$

$\mathcal{D}$ base

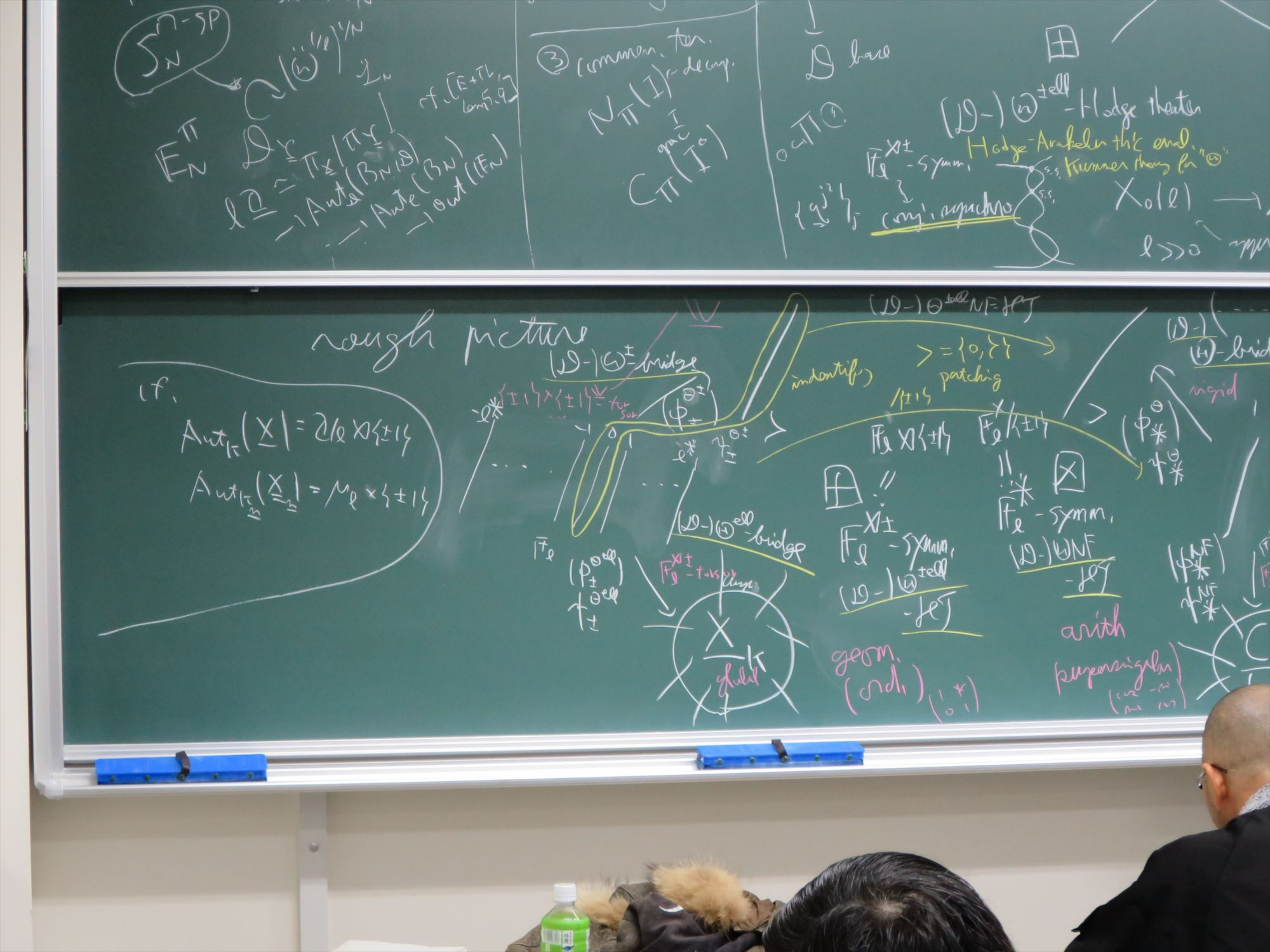
rough story

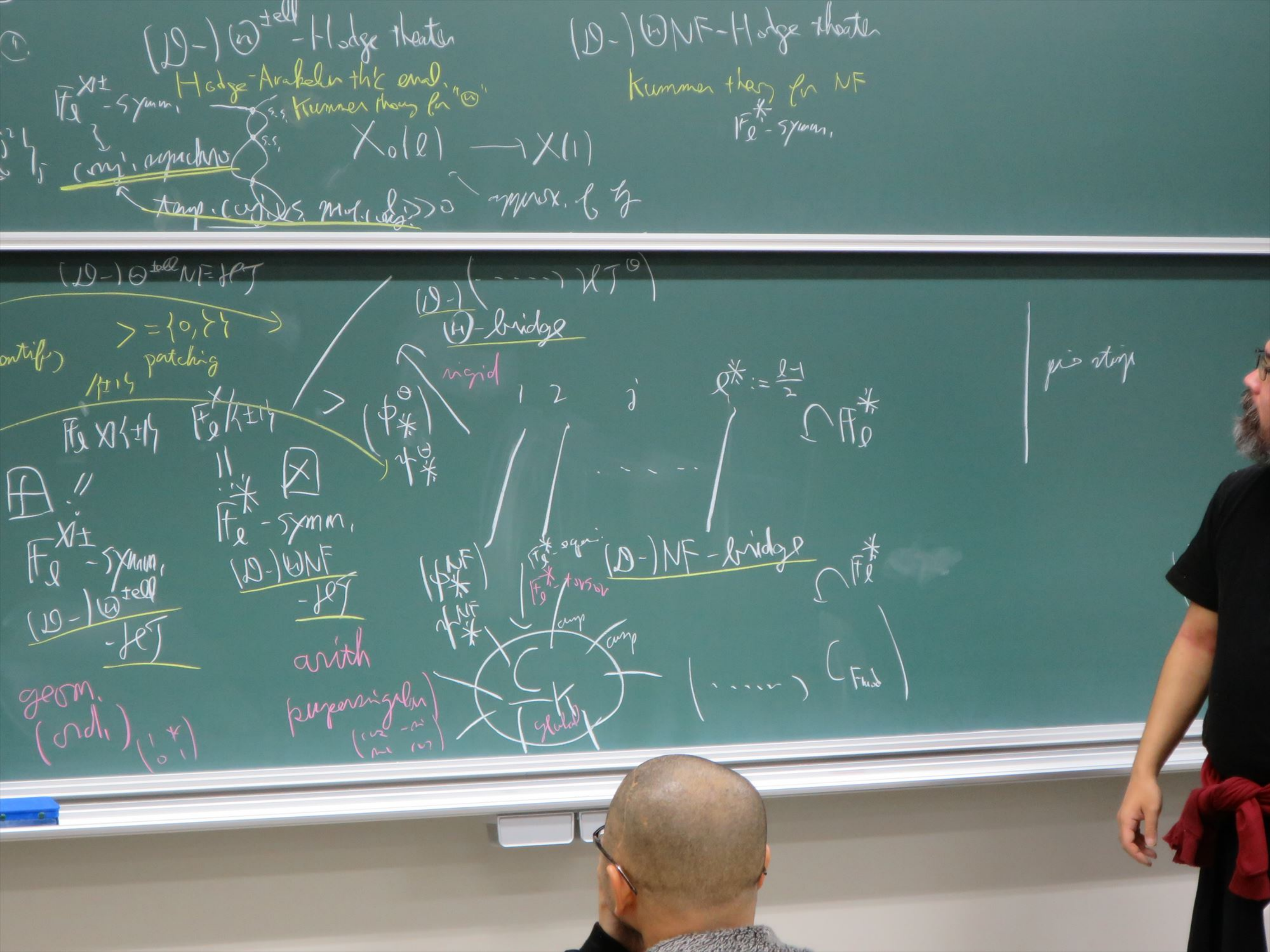
$$\begin{tikzcd}\mathfrak{F} \arrow[d, dash] \\ \mathfrak{D} \text{ base}\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \text{str. of } ((\mathcal{D}\text{-})\Theta^{\pm \mathrm{ell}}\text{NF-Hodge theater} \arrow[dl, dash] \arrow[dr, dash] & \\ \boxplus & & \boxtimes\end{tikzcd}$$

$(\mathcal{D}\text{-})\Theta^{\pm \mathrm{ell}}$-Hodge theater

$(\mathcal{D}\text{-})\Theta$NF-Hodge theater

③ commen. ter.
N_π(I) r-decomp.
C_π(I°)
𝔇 base
(𝔇-) Θ^{±ell}-Hodge theater
Hodge-Arakelov th'c eval.
Kummer theory for "Θ"
F_l^{⋊±}-symm.
conj. synchro
X_0(l)
l >>> 0
cf. [E+Th, Lem 5.9]
Out(F_N)
Aut(B_N)
rough picture
cf.
Aut_k(X) = Z/l ⋊ {±1}
Aut_k(X) = μ_l × {±1}
(𝔇-) Θ^±-bridge
l^* {±1} ⋀ {±1} -tor sor
(φ_±^{Θ^±})
ψ_±^{Θ^±}
(𝔇-) Θ^{ell}-bridge
F_l
(φ_±^{Θ^{ell}})
ψ_±^{Θ^{ell}}
F_l^{⋊±}-torsor
X
K
global
identify
(𝔇-) Θ^{±ell} NF-HT
> = {0, 3}
patching
/{±1}
F_l ⋊ {±1}
F_l^× / {±1}
F_l^{⋊±}-symm.
(𝔇-) Θ^{±ell}-HT
F_l^*-symm.
(𝔇-) Θ NF-HT
geom.
(ind.)
(1 * / 0 1)
arith
(superisingular)
(𝔇-)
Θ-bridge
rigid
(φ_*^Θ)
ψ_*^Θ
(φ_*^{NF})
ψ_*^{NF}



① $(\mathcal{D}\text{-})\,\Theta^{\pm \mathrm{ell}}$-Hodge theater

$(\mathcal{D}\text{-})\,\Theta\mathrm{NF}$-Hodge theater

Hodge-Arakelov th'c eval, Kummer theory for "Θ"

Kummer theory for NF

$\mathbb{F}_l^{\rtimes\pm}$-symm.

$\mathbb{F}_l^{*}$-symm.

conj. synchro

$X_0(l) \longrightarrow X(1)$

approx.

Temp. (cyclic) s. mod. deg. >> 0

$(\mathcal{D}\text{-})\,\Theta^{\pm \mathrm{ell}}\mathrm{NF}$-HT

$(\ldots\ldots, \mathcal{HT}^{\Theta})$

$(\mathcal{D}\text{-})$ Θ-bridge

rigid

$> = \{0, \zeta\}$

patching

ontif.)

$/\{\pm 1\}$

$\mathbb{F}_l \rtimes \{\pm 1\}$ $\quad \mathbb{F}_l^{\rtimes}/\{\pm 1\}$ $\quad > \quad (\phi^{\Theta}_{*}, \psi^{\Theta}_{*})$

1 2 j $l^* := \frac{l-1}{2}$ $\quad \curvearrowleft \mathbb{F}_l^{*}$

$\mathbb{F}_l^{\rtimes\pm}$-symm.

$\mathbb{F}_l^{*}$-symm.

$(\mathcal{D}\text{-})\,\Theta^{\pm \mathrm{ell}}$-HT

$(\mathcal{D}\text{-})\,\Theta\mathrm{NF}$-HT

geom. (cusp) $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

arith (supersingular)

$(\phi^{\mathrm{NF}}_{*}, \psi^{\mathrm{NF}}_{*})$

$\mathbb{F}_l^{*}$-equi.

$\mathbb{F}_l^{*}$-torsor

$(\mathcal{D}\text{-})$ NF-bridge

$\curvearrowleft \mathbb{F}_l^{*}$

cusp cusp

C_K global

$(\ldots\ldots, C_{\mathrm{Fmod}})$

prime strips

rough story

str. of $(\mathcal{D}\text{-})\Theta^{\pm ell}$NF-Hodge theater

⊞ ⊠

$\mathbb{F}$
$\downarrow$
$\mathcal{D}$ base

$\circ \curvearrowright \Pi$ ①

$(\mathcal{D}\text{-})\Theta^{\pm ell}$-Hodge theater

Hodge-Arakelov th'c eval.

Kummer theory for "Θ"

$\mathbb{F}_l^{\rtimes\pm}$-symm.

$\{q^{j^2}\}_j$

conj. synchro

s.s.

s.s.

temp. (cusps. mult. deg.) > 0

$X_0(l) \longrightarrow X(1)$

approx. by

$(\mathcal{D}\text{-})\Theta$NF-Hodge theater

Kummer theory for NF

$\mathbb{F}_l^{*}$-symm.

$F_{mod} \rightsquigarrow$ ring str. $\left\{ \begin{array}{l} ⊠\text{-line bdl} \\ ⊞\text{-line bdl} \end{array} \right.$

ten.

-decomp.

$(\mathcal{D}\text{-})\Theta^{\pm ell}$NF-HT

$(\ldots\ldots)$ HT$^{\Theta}$

$> = \{0, \succ\}$

identify

patching

$\{\pm 1\}$

bridge

$(\phi_{\pm}^{\Theta^{\pm}})$

$\mathbb{F}_l \rtimes \{\pm 1\}$

$\mathbb{F}_l^{\rtimes}/\{\pm 1\}$

$> (\phi_{*}^{\Theta})$

ψ_*^{Θ}

$(\mathcal{D}\text{-})$

Θ-bridge

rigid

1 2 j

$l^* := \frac{l-1}{2}$

$\curvearrowright \mathbb{F}_l^{*}$

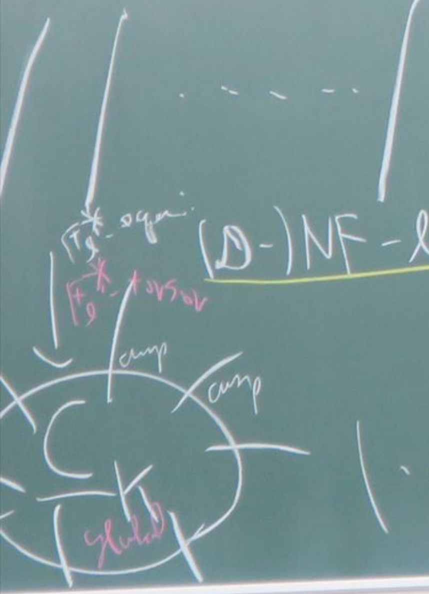
prime strips

$(\cdots\cdots \to \mathcal{HT}^{\Theta})$

bridge

1 2 j $l^{*} := \frac{l-1}{2}$

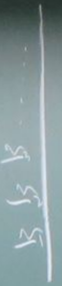
$\curvearrowright \mathbb{F}_l^{*}$



$(D\text{-})$NF-bridge

$\curvearrowright \mathbb{F}_l^{*}$

$(\cdots\cdots, \mathcal{C}_{F_{\mathrm{mod}}})$



prime strips

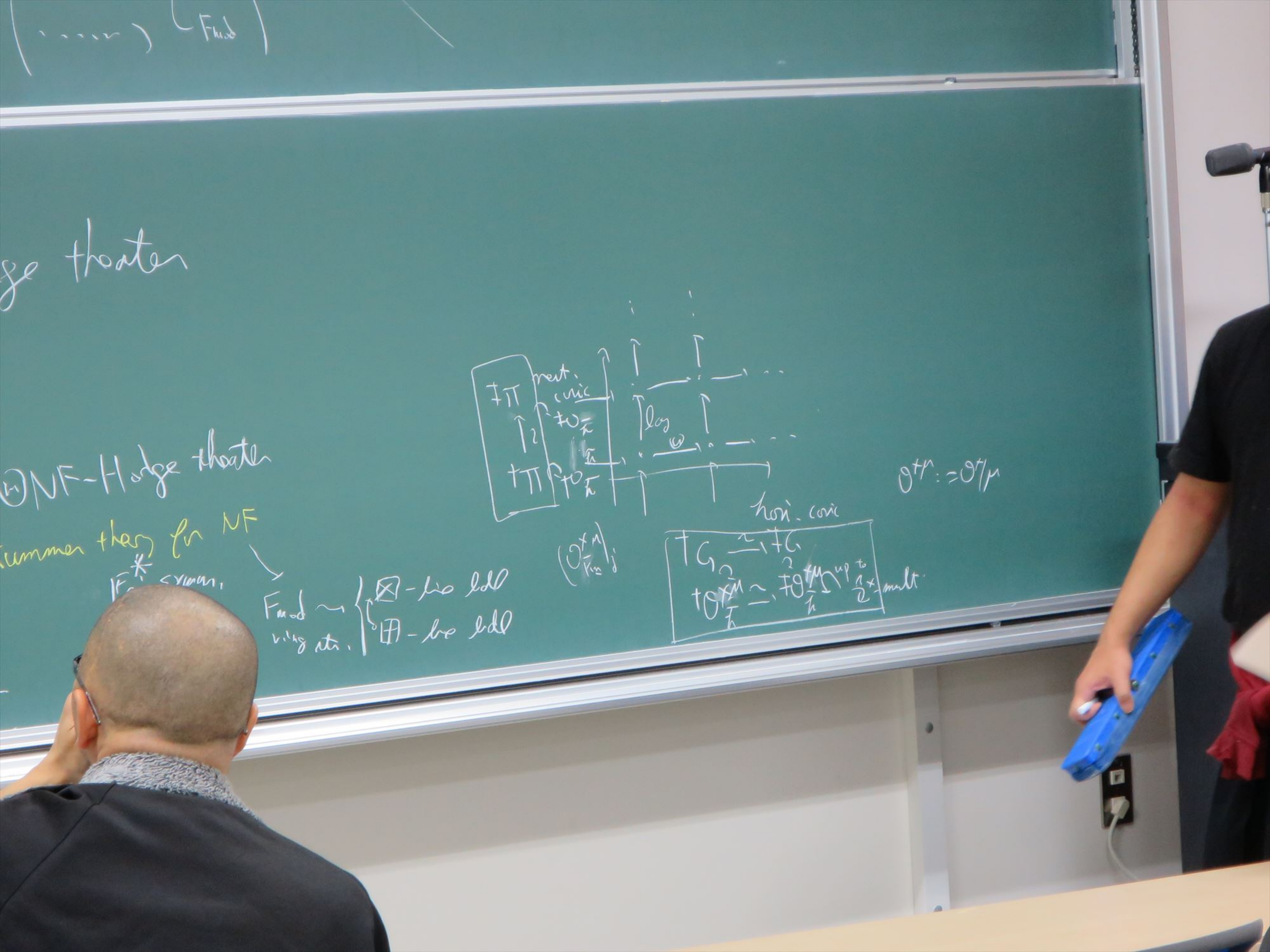
$$\underbrace{\mathcal{F}_0 \cong \mathcal{F}_{>}}_{\downarrow}$$

$\mathcal{F}_{>}$

$$\left(\begin{array}{l} \Pi_X \cong \boxed{{}^{\dagger}\Pi} \\ G_k \cong \boxed{{}^{\dagger}G} \end{array} \right)$$

A

$)^{\pm \mathrm{ell}}$NF-Hodge theater

F_mod
ge theater
NF-Hodge theater
Kummer theory for NF
F_mod ⇝ ring str.
⊠-line bdl
⊞-line bdl
vert. coric
log
hori. coric
up to ±1 mult.
Θ^{×μ} := Θ^×/μ

rough story

$$\bullet \xrightarrow{\Theta\text{-link}} \bullet$$

3-portions of Θ-link

$$\text{local}\begin{cases}\bullet\ \text{unit gp}\\ \bullet\ \text{value gp}\end{cases}$$

- gl. real'd

3-portions of ⊖-link

local
- unit gp
- value gp
- gl. real'd

$$^{\dagger}\mathcal{G} \xrightarrow{\sim} {}^{\ddagger}\mathcal{G}$$

up to $\hat{\mathbb{Z}}^{\times}$-mult.

$$^{\dagger}\mathcal{O}^{\times\mu} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\times\mu}$$

$$^{\dagger}\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \xrightarrow{\sim} {}^{\ddagger}\underline{\underline{q}}^{\mathbb{N}}$$

$$\{\mathbb{R}_{\geq 0}\log(\underline{\underline{q}}^{j^2})_j\}_{v\in\underline{\mathbb{V}}} \xrightarrow{\sim} \{\mathbb{R}_{\geq 0}\log\underline{\underline{q}}\}_{v\in\underline{\mathbb{V}}}$$

fr.d. symbol w/ prod. str.

$\cong \mathbb{R}_{\geq 0}$

$(\mathcal{O}^{\triangleright}/\mathcal{O}^{\times})^{\mathrm{gp}} \otimes \mathbb{R}_{\geq 0}$

$\mathbb{R}_{\geq 0}$

$\underline{\mathbb{V}}$

$\underline{\Theta} \underset{\mathrm{end}}{\curvearrowright} \{\underline{\underline{q}}^{j^2}\}$

$(\circledcirc)^{\mathbb{N}} \curvearrowright \prod$

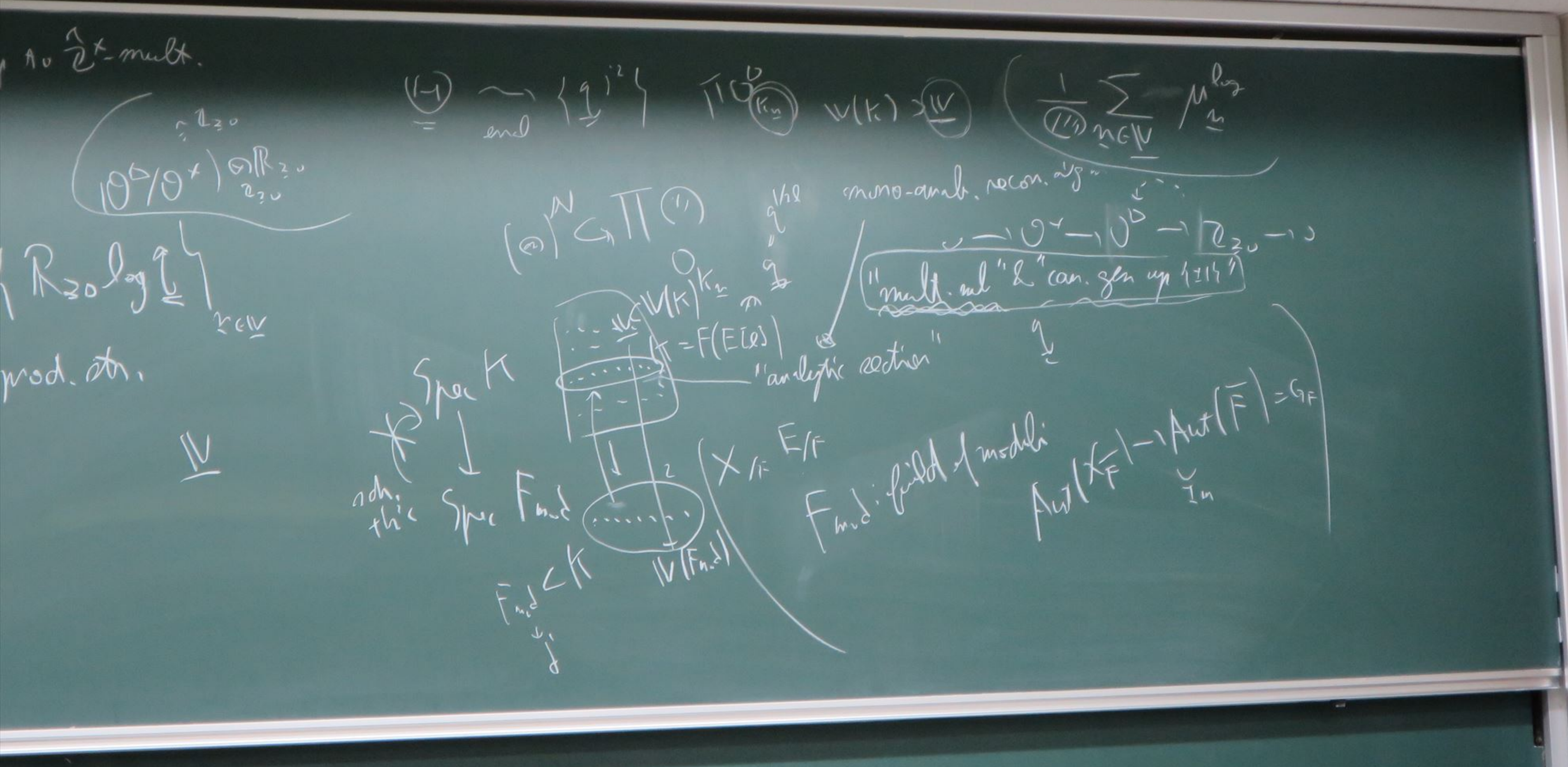
$$\begin{tikzcd} \mathrm{Spec}\,K \arrow[d] \\ \mathrm{Spec}\,F_{\mathrm{mod}} \arrow[u, bend left, "\times"] \end{tikzcd}$$

adh.
thic

$F_{\mathrm{mod}} \subset K$

$\downarrow$

j



A_∞ $\hat{\mathbb{Z}}^\times$-mult.

$\left(\mathcal{O}^\triangleright/\mathcal{O}^\times\right)^{\mathbb{Z}_{\geq 0}} \otimes \mathbb{R}_{\geq 0} \quad \mathbb{Z}_{\geq 0}$

$\{\mathbb{R}_{\geq 0} \log \underline{\underline{q}}\}_{\underline{v} \in \underline{\mathbb{V}}}$

prod. str.

$\underline{\mathbb{V}}$

$\underset{=}{(\text{l-l})} \xrightarrow[\text{end}]{\sim} \{\underline{q}^{j^2}\}$ $\quad \prod \mathcal{O}^\triangleright_{K_{\underline{v}}} \quad \mathbb{V}(K) \supseteq \underline{\mathbb{V}}$

$\left(\frac{1}{(l^*)} \sum_{\underline{v}\in\underline{\mathbb{V}}} \mu^{\log}_{\underline{v}}\right)$

$(\circledast)^N \curvearrowright \prod (\cdot)$

$q^{1/2l}$

$\underline{\underline{q}}$

"mono-anab. recon. alg"

$0 \to \mathcal{O}^\times \to \mathcal{O}^\triangleright \to \mathbb{Z}_{\geq 0} \to 0$

"mult. mod" & "can. gen up $\{\pm 1\}$"

$\underline{\underline{q}}$

$\mathcal{O}_{K_{\underline{v}}}$ $\quad \mathbb{V}(K)$

$K = F(E[l])$

"analytic section"

$F_{\mathrm{mod}} \subset K$

F_{mod}

$X_{/F} \quad E_{/F}$

F_{mod}: field of moduli

$\mathrm{Aut}(X_F) \to \mathrm{Aut}(\overline{F}) = G_F$

Im

3-portions of Θ-link

local { • unit gp
local { • value gp

• gl. real'd

$^{\dagger}\mathcal{G} \xrightarrow{\sim} {}^{\ddagger}\mathcal{G}$ ⟲ ⟲

up to $\hat{\mathbb{Z}}^{\times}$-mult.

$^{\dagger}\mathcal{O}^{\times\mu} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\times\mu}$

$^{\dagger}\{\underline{\underline{q}}^{j^2}\}^{\mathbb{N}} \xrightarrow{\sim} {}^{\ddagger}\underline{\underline{q}}^{\mathbb{N}}$

$\{\mathbb{R}_{\geq 0}\log(\underline{\underline{q}}^{j^2})_j\}_{v\in\underline{\mathbb{V}}} \xrightarrow{\sim} \{\mathbb{R}_{\geq 0}\log\underline{\underline{q}}\}_{v\in\underline{\mathbb{V}}}$

$(\mathcal{O}^{\triangleright}/\mathcal{O}^{\times})^{\cong \mathbb{Z}_{\geq 0}} \otimes \underset{\mathbb{Z}_{\geq 0}}{\mathbb{R}_{\geq 0}}$

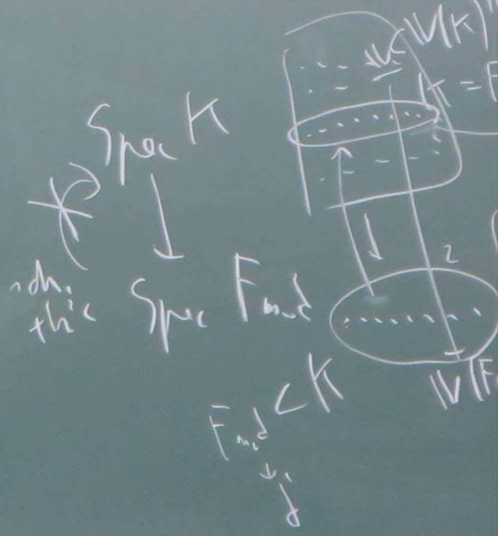
frd symbol w/ mod. str.

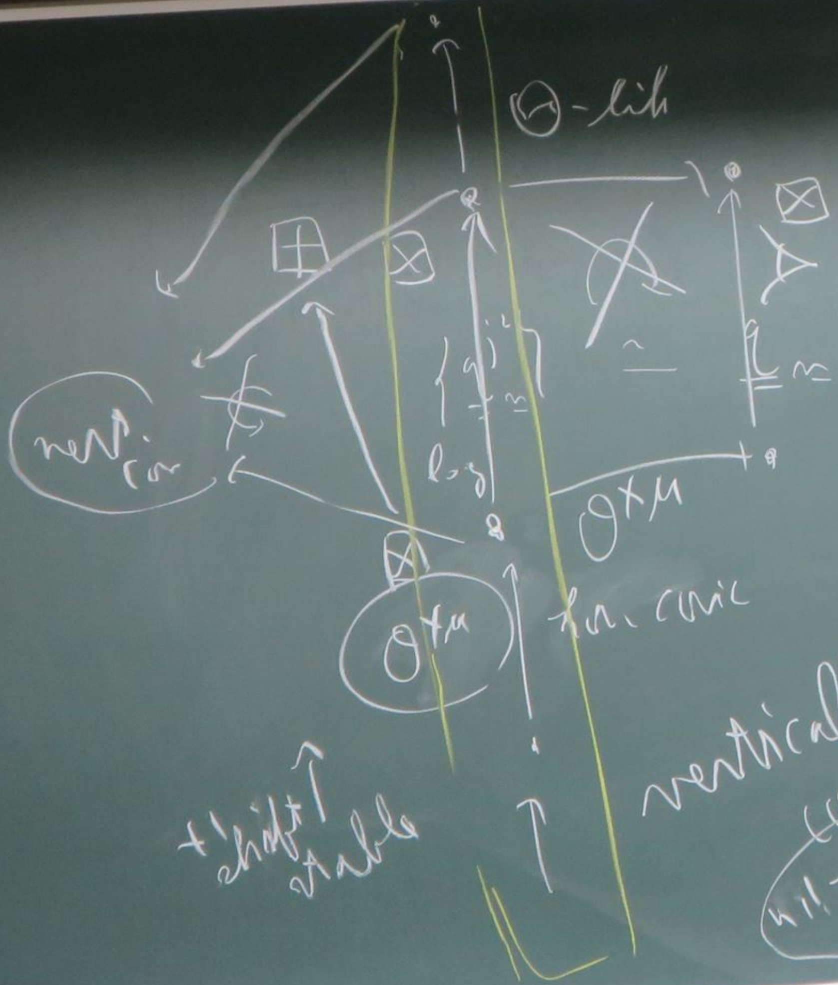
$\underline{\underline{\Theta}} \underset{\text{eval}}{\leadsto} \{\underline{\underline{q}}^{j^2}\}$

$(\Theta)^{\mathbb{N}} \curvearrowright \prod (\Theta)$

Want to protect { real. gp portion / gl val'd portion } from $\hat{\mathbb{Z}}^{\times}$-indet

mild indet ⟵





- $\overset{\boxtimes}{\mathcal{O}^{\times\mu}}$ is shared

Want to protect { ral.
gl

⤳ by using log-link, we get "⊞" obj.

⤳ But log-Θ-lattice is highly non-commutative

⤳ use vertical lino

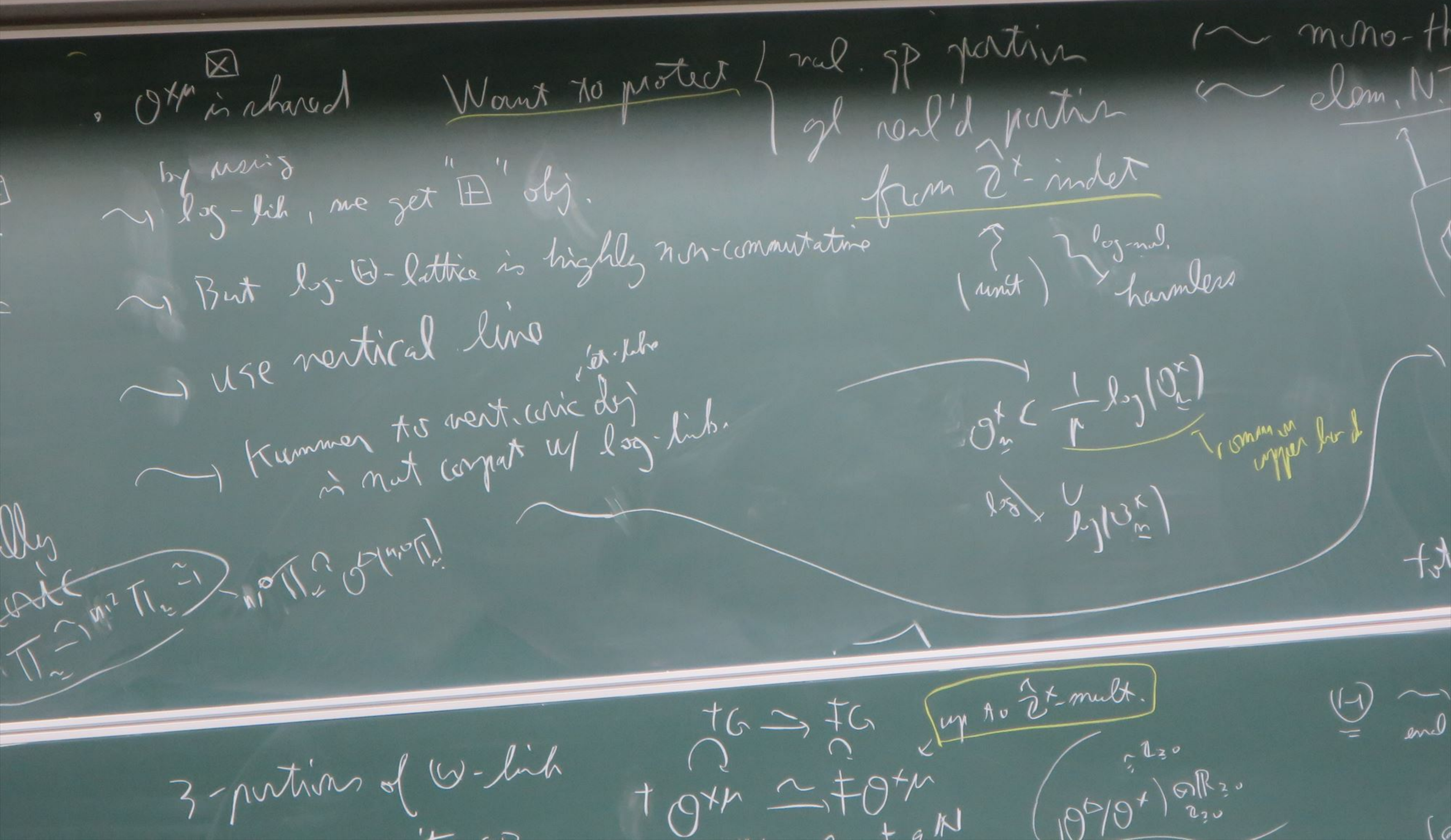
⤳ Kummer to vert. coric obj is not compat w/ log-link.
ét-like

vertically coric

${}^{n,1}\Pi_{\sim} \xrightarrow{\sim} {}^{n,2}\Pi_{\sim} \xrightarrow{\sim} \dots \rightarrow {}^{n,\circ}\Pi_{\sim} \curvearrowright \mathcal{O}^{\triangleright}({}^{n,\circ}\Pi_{\sim})$

3-portions of Θ-link

$$\begin{tikzcd}
{}^{\dagger}G \arrow[r] \arrow[d, phantom, "\curvearrowright"] & {}^{\ddagger}G \arrow[d, phantom, "\curvearrowright"] \\
{}^{\dagger}\mathcal{O}^{\times\mu} \arrow[r, "\sim"] & {}^{\ddagger}
\end{tikzcd}$$



- $\mathcal{O}^{\times\mu}$ is shared ⊠

Want to protect $\left\{ \begin{array}{l} \text{rad. gp portion} \\ \text{gl norl'd portion} \end{array} \right.$ from $\hat{\mathbb{Z}}^{\times}$-indet

(unit) ↑ ; log-vol. ↘ harmless

(~ mono-th

(~ elem. N

$\rightsquigarrow$ by using log-link, we get "⊞" obj.

$\rightsquigarrow$ But log-Θ-lattice is highly non-commutative

$\rightsquigarrow$ use vertical line

$\rightsquigarrow$ Kummer to vert. coric obj (ét.-like) is not compat w/ log-link.

$$\mathcal{O}^{\times}_{\underline{v}} \subset \underbrace{\frac{1}{p}\log(\mathcal{O}^{\times}_{\underline{v}})}$$

common upper bound

$$\log \searrow \quad \cup \quad \log(\mathcal{O}^{\times}_{\underline{v}})$$

lly

~~ic~~

$$\Pi_{\underline{v}} \xrightarrow{\sim} {}^{n,2}\Pi_{\underline{v}} \xrightarrow{\sim} {}^{n,\circ}\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}({}^{n,\circ}\Pi_{\underline{v}})$$

3-portions of Θ-link

$$\begin{tikzcd} {}^{\dagger}G \arrow[r, "\sim"] & {}^{\ddagger}G \\ {}^{\dagger}\mathcal{O}^{\times\mu} \arrow[u, phantom, "\curvearrowright"] \arrow[r, "\sim"] & {}^{\ddagger}\mathcal{O}^{\times\mu} \arrow[u, phantom, "\curvearrowright"] \end{tikzcd}$$

up to $\hat{\mathbb{Z}}^{\times}$-mult.

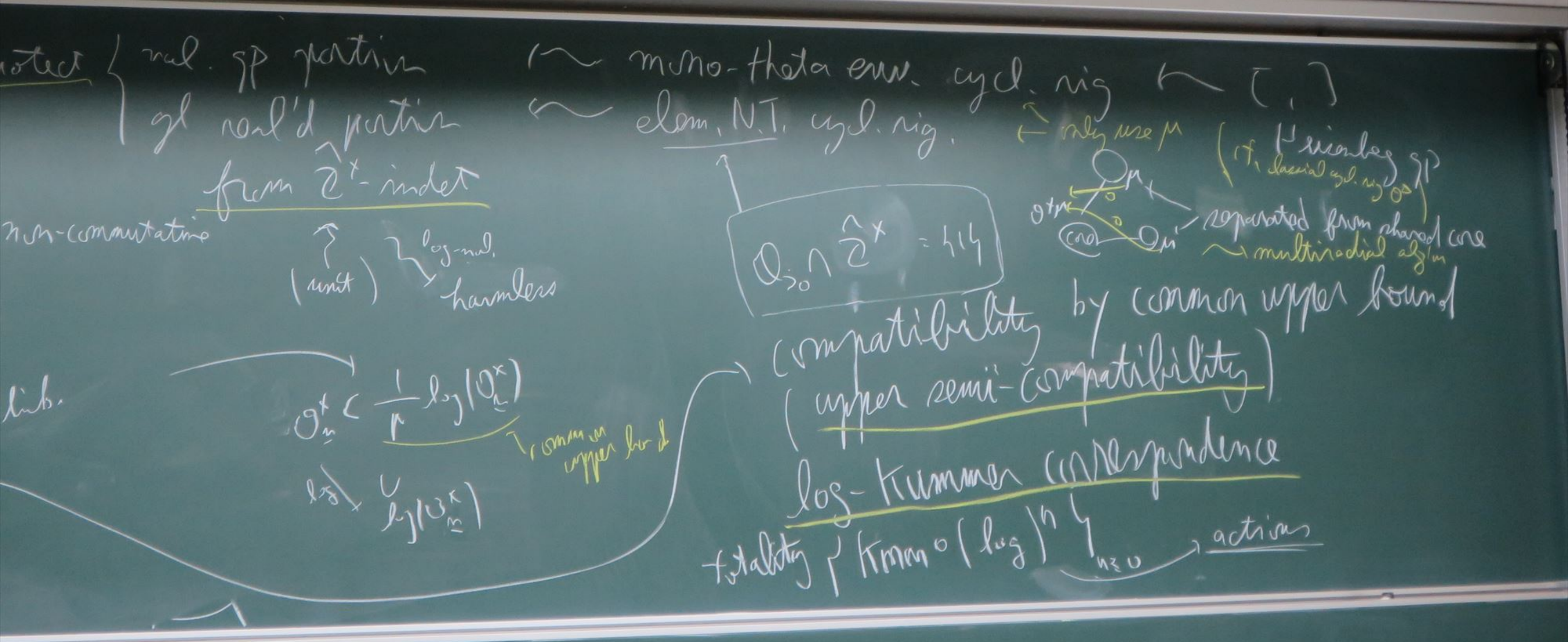
$\mathbb{N}$

$$(\mathcal{O}^{\triangleright}/\mathcal{O}^{\times})^{gp} \otimes \mathbb{R}_{\geq 0} \qquad \mathbb{Z}_{\geq 0} \qquad \mathbb{Z}_{\geq 0}$$

(Θ) ≅

~ end

tot



otect { rad. gp portion ⟵ mono-theta env. cycl. rig ⟵ [,]

gl norm'd portion ⟵ elem. N.T. cycl. rig.

from $\hat{\mathbb{Z}}^\times$-indet

non-commutative

(unit)

log-rad.

harmless

$\boxed{\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}}$

⟵ only use μ

(cf. classical cycl. rig $\mathcal{O}^\triangleright$) μ-invariance gp

$\mathcal{O}^{\times\mu}$

$\mathcal{O}_\mu$

core

$\mathcal{O}^\times_\mu$

separated from shared core

⟶ multiradial alg'm

lib.

$\mathcal{O}^\times_{\underline{v}} \subset \frac{1}{p}\log(\mathcal{O}^\times_{\underline{v}})$

common upper bound

$\log \searrow \quad \cup$

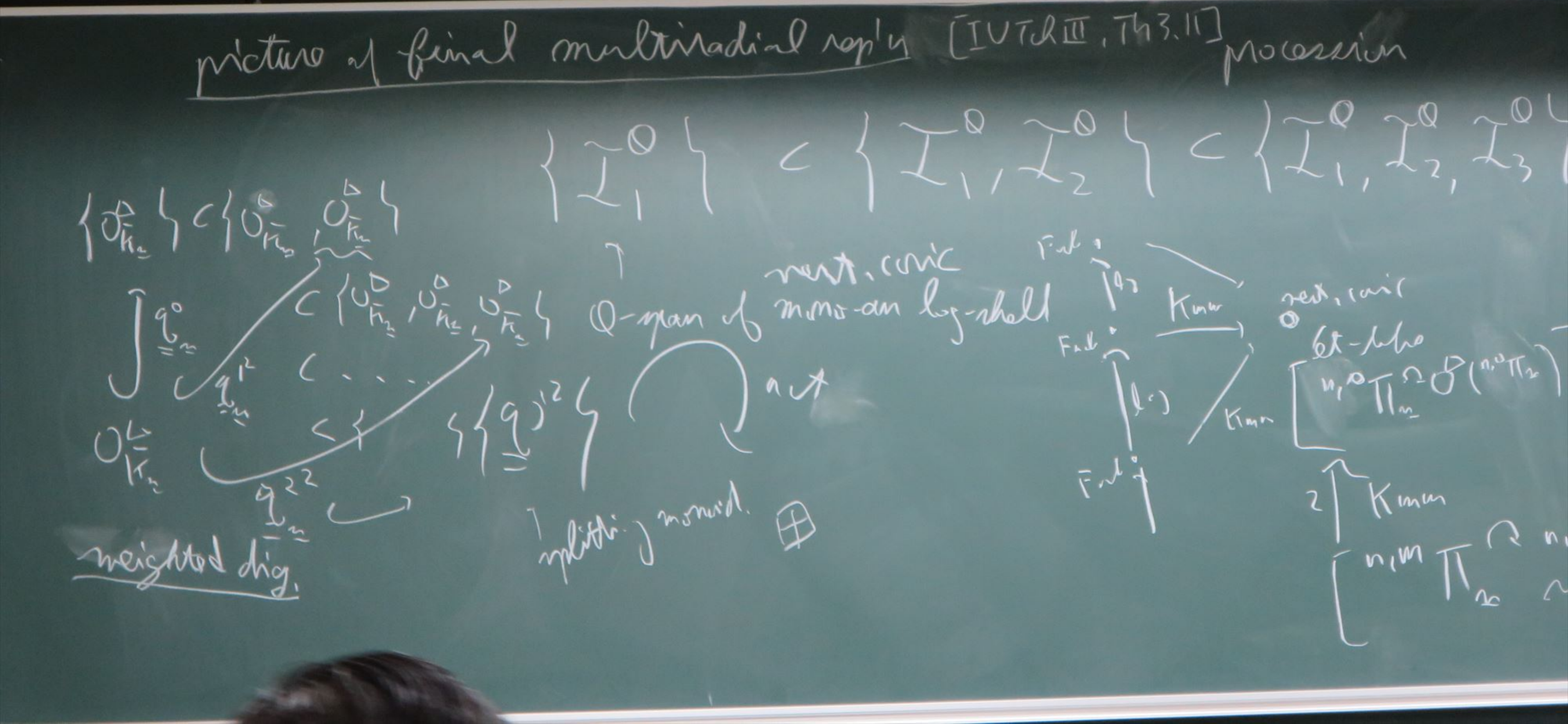
$\log(\mathcal{O}^\times_{\underline{v}})$

compatibility by common upper bound

(upper semi-compatibility)

log-Kummer correspondence

totality $\{\mathrm{Kmm} \circ (\log)^n\}_{n \geq 0}$ ⟶ actions



picture of final multiradial rep'n [IUTchIII, Th3.11] procession

$\{\mathcal{I}^{\mathbb{Q}}_1\} \subset \{\mathcal{I}^{\mathbb{Q}}_1, \mathcal{I}^{\mathbb{Q}}_2\} \subset \{\mathcal{I}^{\mathbb{Q}}_1, \mathcal{I}^{\mathbb{Q}}_2, \mathcal{I}^{\mathbb{Q}}_3$

$\{\mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}\} \subset \{\mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}, \mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}\} \subset \{\mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}, \mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}, \mathcal{O}^{\triangleright}_{\overline{K}_{\underline{v}}}\} \subset \cdots \subset \{$

$\underline{\underline{q}}^{0}_{\underline{v}}$ $\underline{\underline{q}}^{1^2}_{\underline{v}}$ $\underline{\underline{q}}^{2^2}_{\underline{v}}$

$\mathcal{O}^{\perp}_{\overline{K}_{\underline{v}}}$

weighted dig.

$\{\{\underline{\underline{q}}^{j^2}\}\}$ act

$\mathbb{Q}$-span of vert. conic mono-an log-shell

splitting monoid. ⊞

Frob. Frob. Frob.

q_j $\ell(\cdot)$

Kummer Kummer

vert. conic

◎ ét-like

$[{}^{n,\circledcirc}\Pi_{\underline{v}} \curvearrowright \mathcal{O}^{\triangleright}({}^{n,\circledcirc}\Pi_{\underline{v}})$

2 Kummer

$[{}^{n,m}\Pi_{\underline{v}} \curvearrowright {}^{n,}$

[UTch III, Th3.11]

procession

$$, \tilde{\mathcal{I}}_2^{\mathbb{Q}} \} \subset \{ \mathcal{I}_1^{\mathbb{Q}}, \mathcal{I}_2^{\mathbb{Q}}, \mathcal{I}_3^{\mathbb{Q}} \} \subset \cdots \subset \{ \mathcal{I}_1^{\mathbb{Q}}, \dots, \mathcal{I}_{l^*}^{\mathbb{Q}} \}$$

④ ← NF

shell

$$\begin{tikzcd}
\mathrm{Frd}^{\circ} \arrow[d, "\log"] \arrow[rd] & \\
\mathrm{Frd}^{\circ} \arrow[d, "\log"] \arrow[r, "\mathrm{Kmm}"] & \Theta \ \text{rest, cair} \ \text{ét-like} \left[{}^{n,\circ}\Pi_v \curvearrowright \mathcal{O}^{\triangleright}({}^{n,\circ}\Pi_v) \right] \\
\mathrm{Frd}^{\circ} \arrow[d] \arrow[ru, "\mathrm{Kmm}"'] & \left[{}^{n,m}\Pi_v \curvearrowright \underbrace{{}^{n,m}\mathcal{O}^{\triangleright}}_{\mathrm{Frd},\ m\in\mathbb{Z}} \right] \arrow[u, "\wr"', "\mathrm{Kmm}"] \\
\ &
\end{tikzcd}$$

$$\curvearrowright \left\{ \left(F_{\mathrm{mod}}^{\times} \right)_j \right\}_{j=1,\dots,l^*}$$

$\boxed{1\times}$

- log-volume is compat

$\left({}^{\mathsf{T}}A \overset{\mathrm{bij}}{\xrightarrow{\sim}} \log \right.$

$\mu^{\log}(A)$

$\ldots, \mathcal{I}^{\mathbb{Q}}_{\ell^*} \}$

④ $\hookrightarrow$ NF

$$0 \to \mathcal{O}^{\times}_{\underline{k}} \to \mathcal{O}^{\triangleright}_{\underline{k}} \to \mathbb{Z}_{\geq 0} \to 0$$

$A \subset K_{\underline{k}}$

$\{ (\mathcal{F}^{\times}_{\mathrm{mod}})_j \}_{j=1,\ldots,\ell^*}$

$\boxed{\mathrm{IX}}$

- log-volume is compat w/ log-link

$$\left(\forall A \overset{\mathrm{bij}}{\xrightarrow{\sim}} \log(A) \quad \mu^{\log}(A) = \mu^{\log}(\log(A)) \right)$$

$$\begin{tikzcd} \mathcal{F} \arrow[d, squiggly] & \\ \mathcal{D} \arrow[d, "\Theta"'] & \boxed{\text{mert. conic}} \\ \mathcal{D}^{\vdash} & \end{tikzcd}$$

$\vdash$ mono-an.

harmless indet

hol. log-shell

$\downarrow\wr$ an orbit of isom's

mono-an log-shell

non-interference

}

log-Kummer

(Ц) ←— canonical splitting ↩— mono-theta const.

up to μ_{2l}-mult.

Galois eval.

NF ↩— Belyi cusp'zation

(+ ell. cusp'zation)

$F^{\times}_{mod} \cap \prod \mathcal{O}_{\underline{v}} = \mu$

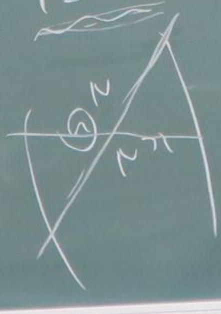
(Gal eval) { Kummer compat. functoriality

eval should come from π_1-thic section

(⟹ eval should come from a secon pt

Sect. Conj.)

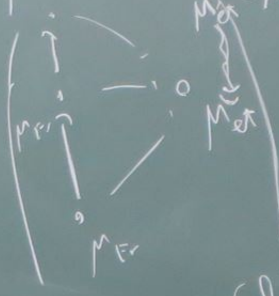
(1-1) ⇝ $\{q^{j^2}\}$



invisible

indet.

[IUTchIII, Rem 3.11.4]



$\mu^{\times}_{et} \cong \mu_{?}$

indet. $(\pm \mathbb{N}_{\geq 1})$

log

loops ∴ cyclotomes

∃ labels

(Ц) ⊞ : theta fct. (trans.)

Gal. eval. : NF : nat. fct. (alg.)

no labels ⊠

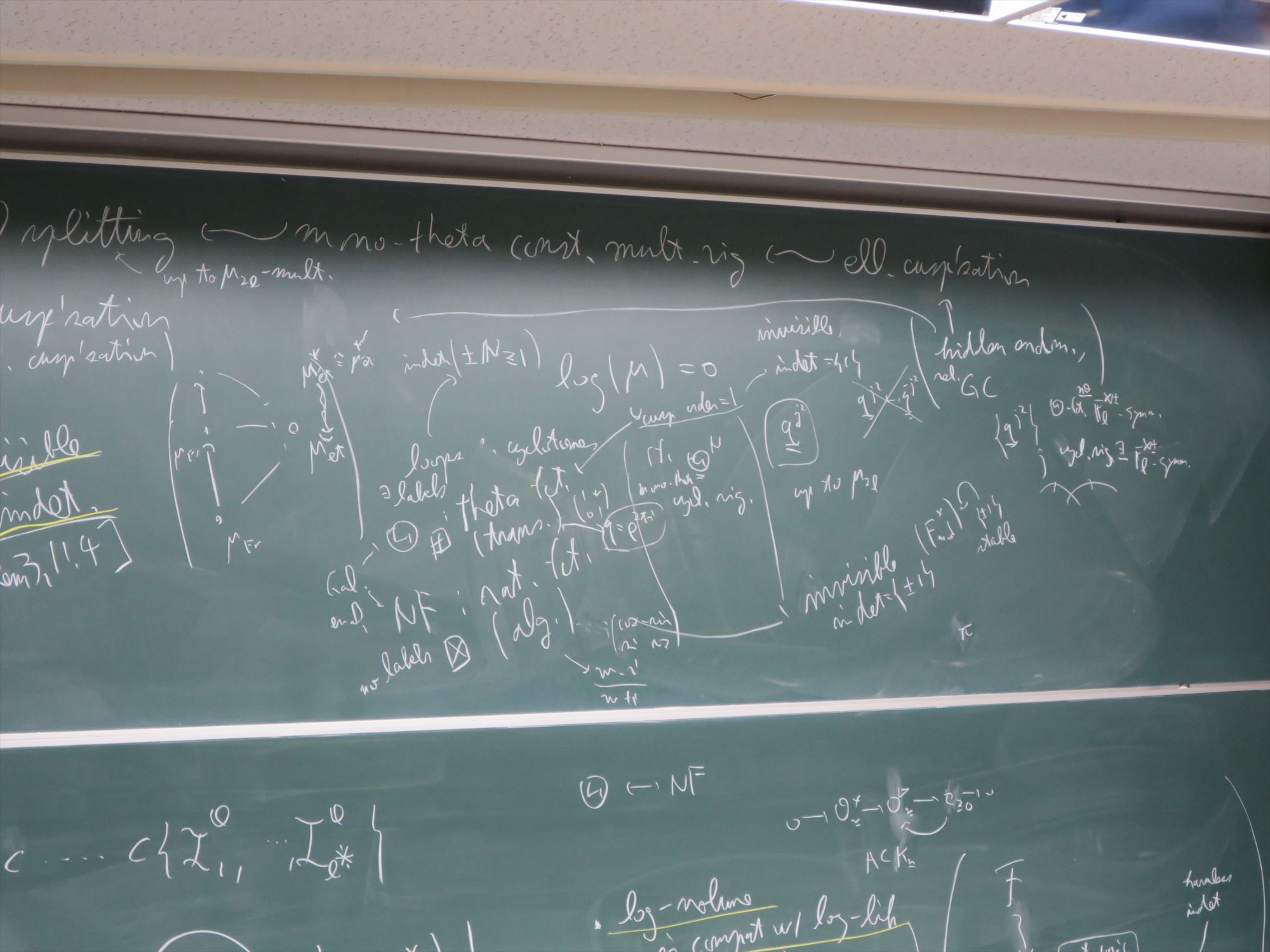
multiradial rep'n [IUTchIII, Th3.11]

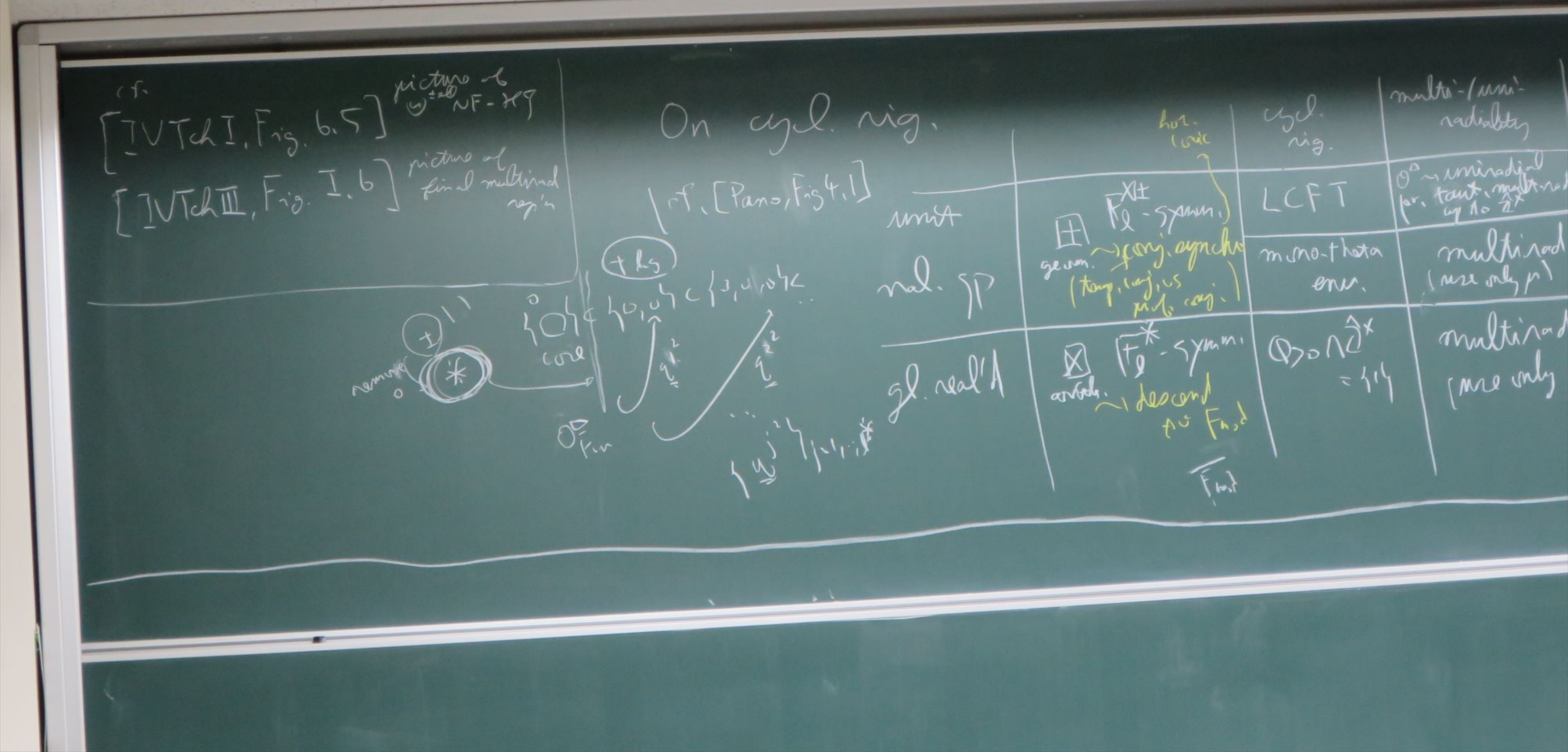
procession

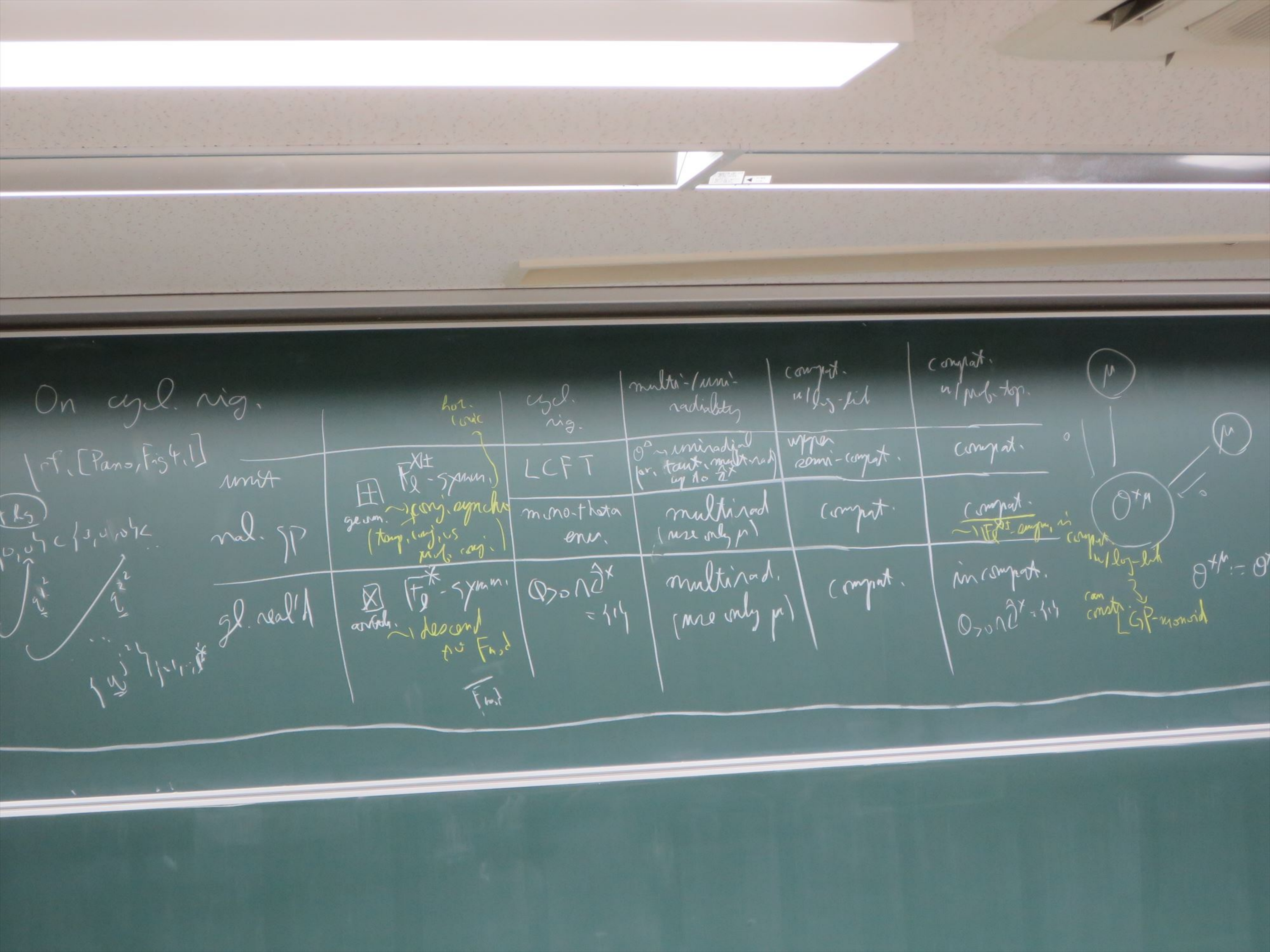
(Ц)

$$\{\mathcal{I}^{\otimes}_1\} \subset \{\mathcal{I}^{\otimes}_1, \mathcal{I}^{\otimes}_2\} \subset \{\mathcal{I}^{\otimes}_1, \mathcal{I}^{\otimes}_2, \mathcal{I}^{\otimes}_3\} \subset \cdots \subset \{\mathcal{I}^{\otimes}_1, \dots, \mathcal{I}^{\otimes}_{l^*}\}$$

$\{(F^{\times}_{mod}$


splitting ⟵ mono-theta const. mult. rig ⟵ ell. cusp'zation
up to μ_{2l}-mult.
cusp'zation
cusp'zation
$\mu^{*}_{et} \cong \mu_{\infty}$
indet. $(\pm\mathbb{N}_{\geq 1})$
$\log(\mu) = 0$
invisible,
indet = {1}
hidden endm.,
rel. GC
μ_{et}
μ_{Fr}
loops cyclotomes
∃ labels
theta fct.
(trans.)
$\begin{pmatrix}1 & * \\ 0 & 1\end{pmatrix}$
$q = e^{2\pi i}$
cusp order = 1
mono-theta = cycl. rig.
q^{j^2}
up to μ_{2l}
$\{q^{j^2}\}$
Θ-fct. $\overline{F}^{\times\pm}_{l}$-symm.
cycl. rig. ∃ $\overline{F}^{\times\pm}_{l}$-symm.
Gal. end.
NF
nat. fct.
(alg.)
no labels
invisible $(F^{\times}_{mod})$ {±1} stable
invisible
indet = {±1}
visible
index
Ⓛ ⟵ NF
$0 \to O^{\times} \to O^{\triangleright} \to \mathbb{R}_{\geq 0} \to 0$
ACK
log-volume
compat w/ log-link
$\{\mathcal{I}^{\mathbb{Q}}_{1}, \ldots, \mathcal{I}^{\mathbb{Q}}_{l^*}\}$
harmless indet





On cycl. rig.

cf. [Pano, Fig 4.1]

$\{0, u\} \subset \{3, 4, 5\} < \dots$

$\underline{q}^{1^2}$

$\underline{q}^{2^2}$

$\dots$

$\{\underline{q}^{j^2}\}_{j=1,\dots,l^*}$

	hor. conic	cycl. rig.	multi-/uni-radiality	compat. w/ log-link	compat. w/ prof. top.
unit	⊞ $\overline{\mathbb{F}}_l^{\rtimes\pm}$-symm. geom. ⇝ {conj. synchro (temp. conj. vs prof. conj.)}	LCFT	Θ^Δ ⇝ uniradial (or, taut. multirad up to $\hat{\mathbb{Z}}^\times$)	upper semi-compat.	compat.
val. gp		mono-theta env.	multirad (use only μ)	compat.	compat. ⇝ $\mathbb{F}_l^{\rtimes\pm}$-symm. is compat. w/ log-link
gl. real'd	⊠ $\overline{\mathbb{F}}_l^{*}$-symm. arith. ⇝ descend to F_{mod} $\overline{F}_{mod}$	$\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$	multirad. (use only μ)	compat.	incompat. $\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^\times = \{1\}$

μ

μ

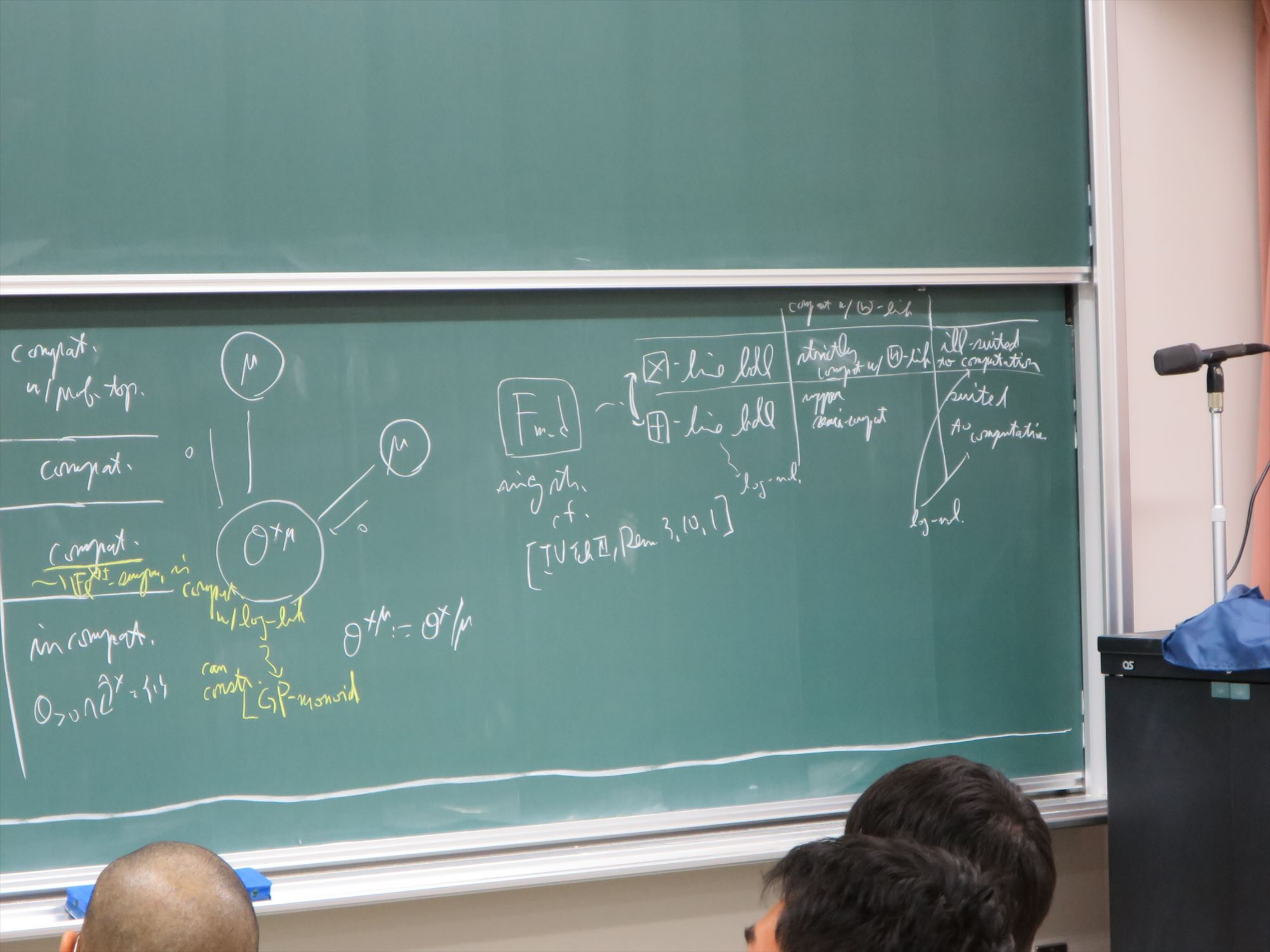
$\circ$

$\circ$

$\Theta^{\times\mu}$

$\Theta^{\times\mu} := \Theta^{\times}$

can constr. ⇝ GP-monoid



compat. w/ prof. top.

compat.

compat.

~ $\mathbb{F}_l^{\rtimes\pm}$-symm, is compat w/ log-link ⇝ GP-monoid

can const.

incompat.

$\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$

$\mathcal{O}^{\times\mu} := \mathcal{O}^{\times}/\mu$

$\boxed{F_{\text{mod}}}$

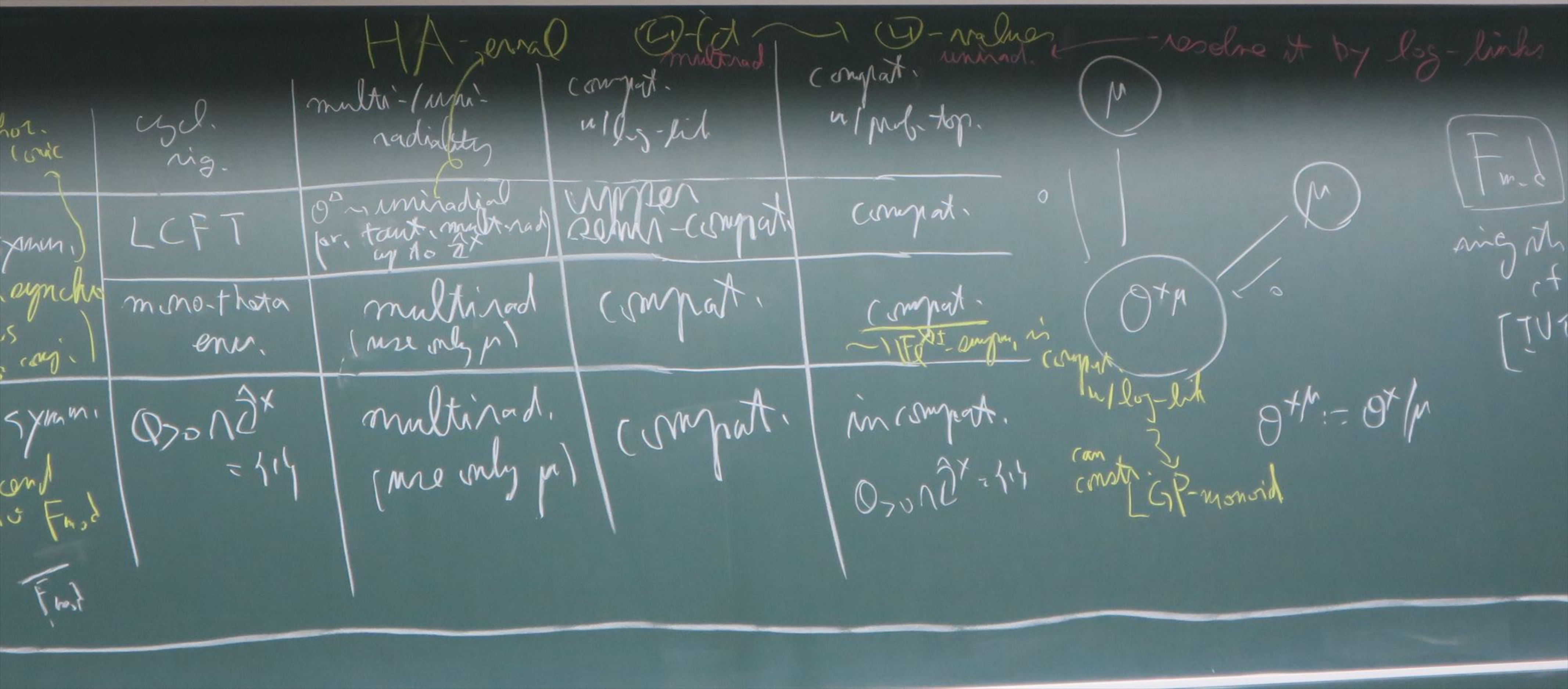
ring th.

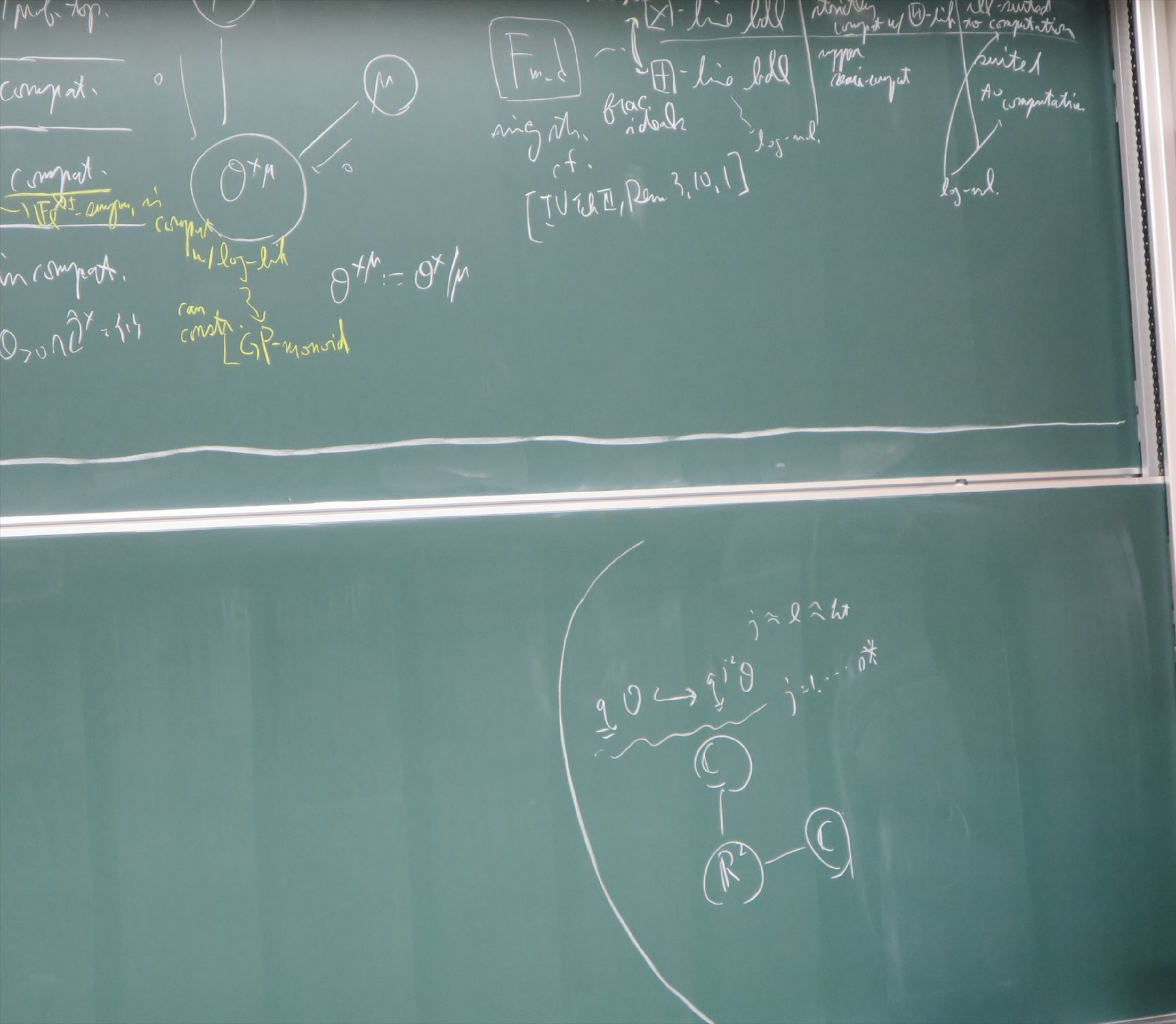
cf. [IUTchII, Rem 3.10.1]

	compat w/ Θ-link	
⊠-line bdl	strictly compat w/ Θ-link	ill-suited to computation
⊞-line bdl	upper semi-compat	suited to computation

log-vol.

log-vol.





mod. top.

compat.

compat.

$\mathbb{F}_\ell^{\rtimes\pm}$-compon. is

incompat.

$\mathcal{O}_{>0} \cap \widehat{\mathcal{O}}^{\times} = \{1\}$

$\mathcal{O}^{\times\mu}$

μ

compat. w/ log-link

can constr. LGP-monoid

$\mathcal{O}^{\times\mu} := \mathcal{O}^{\times}/\mu$

$\boxed{\mathcal{F}_{\mathrm{mod}}}$

sing. str.

cf. [IUTchII, Rem 3.10.1]

frac. ideals

$\boxed{\times}$-line bdl	strictly compat. w/ Θ-link	ill-suited to computation
$\boxed{+}$-line bdl	upper semi-compat	suited to computation

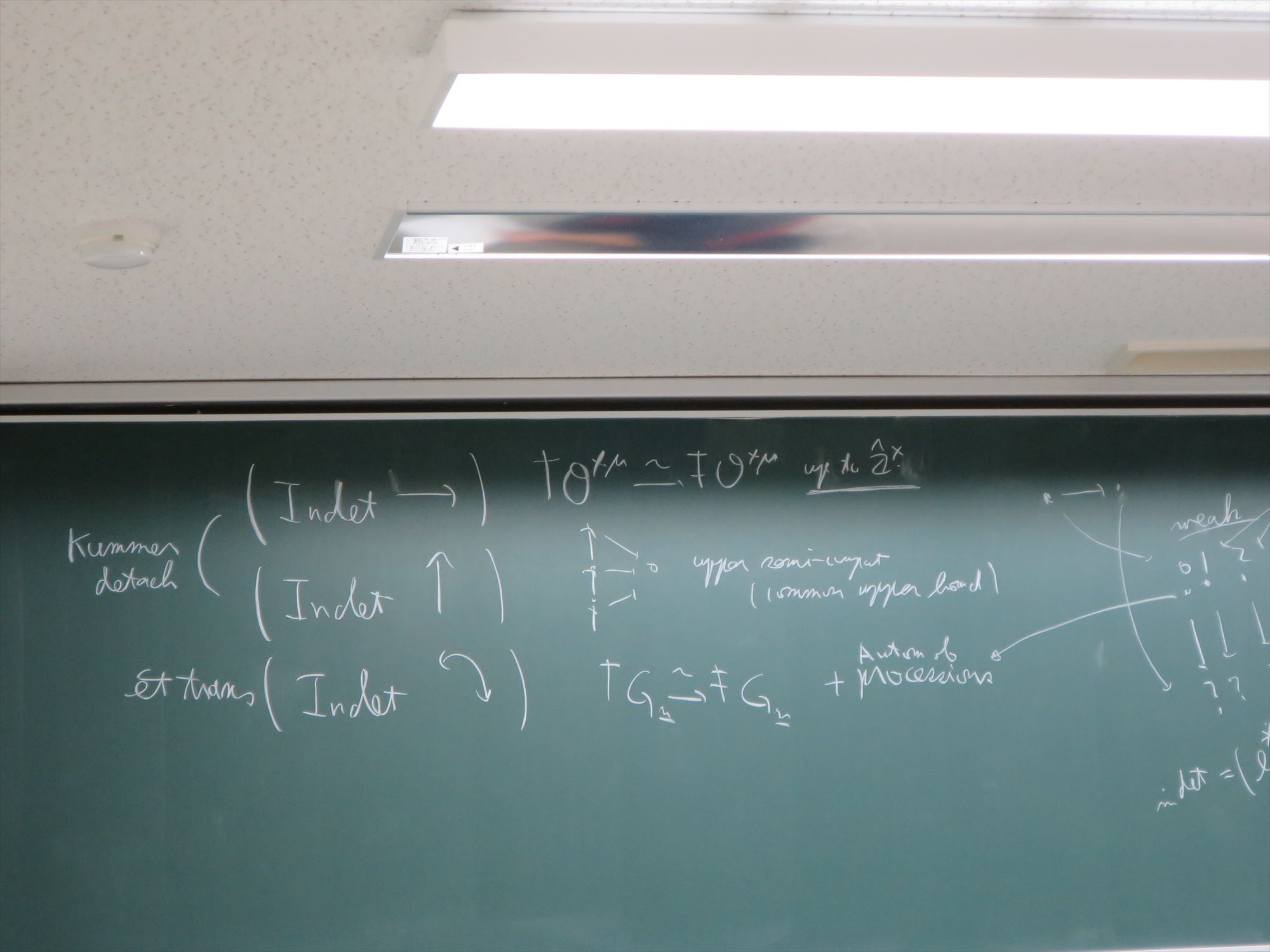
log-vol.

log-vol.

$\underline{\underline{q}}\ \vartheta \hookrightarrow \underline{\underline{q}}^{j^2}\vartheta$

$j = 1, \dots, \ell^*$

$j \approx \ell \approx ht$



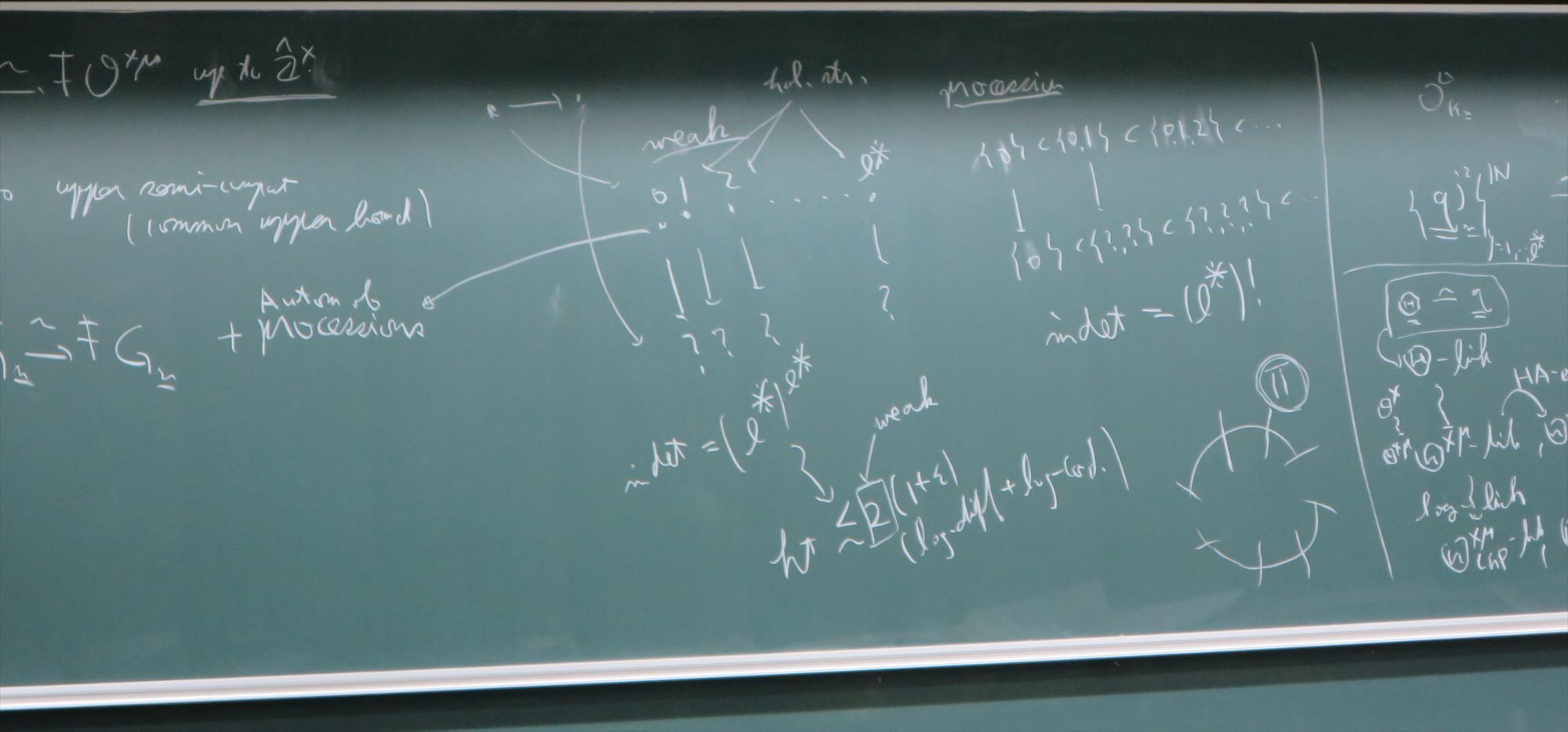
Kummer detach ((Indet →) $^{\dagger}\mathcal{O}^{\times\mu} \xrightarrow{\sim} {}^{\ddagger}\mathcal{O}^{\times\mu}$ up to $\hat{\mathbb{Z}}^{\times}$

(Indet ↑) upper semi-compact (common upper bound)

ét trans (Indet ⤵) $^{\dagger}G_{\underline{v}} \xrightarrow{\sim} {}^{\ddagger}G_{\underline{v}}$ + Autom of processions

weak

indet = (



$\simeq \ddagger\mathcal{O}^{\times\mu}$ up to $\hat{\mathbb{Z}}^{\times}$

upper semi-compat
(common upper bound)

Autom of + processions

$\overset{\sim}{\to} \ddagger G_{\underline{v}}$

hol. str.

weak

$0 \quad \epsilon \quad \cdots \quad \ell^*$

$? \quad ? \quad ? \quad ?$

$\text{indet} = (\ell^*)^{\ell^*}$

$\rightsquigarrow \ \text{ht} \lesssim \boxed{2}(1+\epsilon)(\text{log-diff} + \text{log-cond.})$

weak

procession

$\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \cdots$

$\{0\} \subset \{?,?\} \subset \{?,?,?\} \subset \cdots$

$\text{indet} = (\ell^*)!$

$\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$

$\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j=1,\dots,\ell^*}^{\mathbb{N}}$

$\boxed{\Theta \simeq \underline{\underline{q}}}$

Θ-link

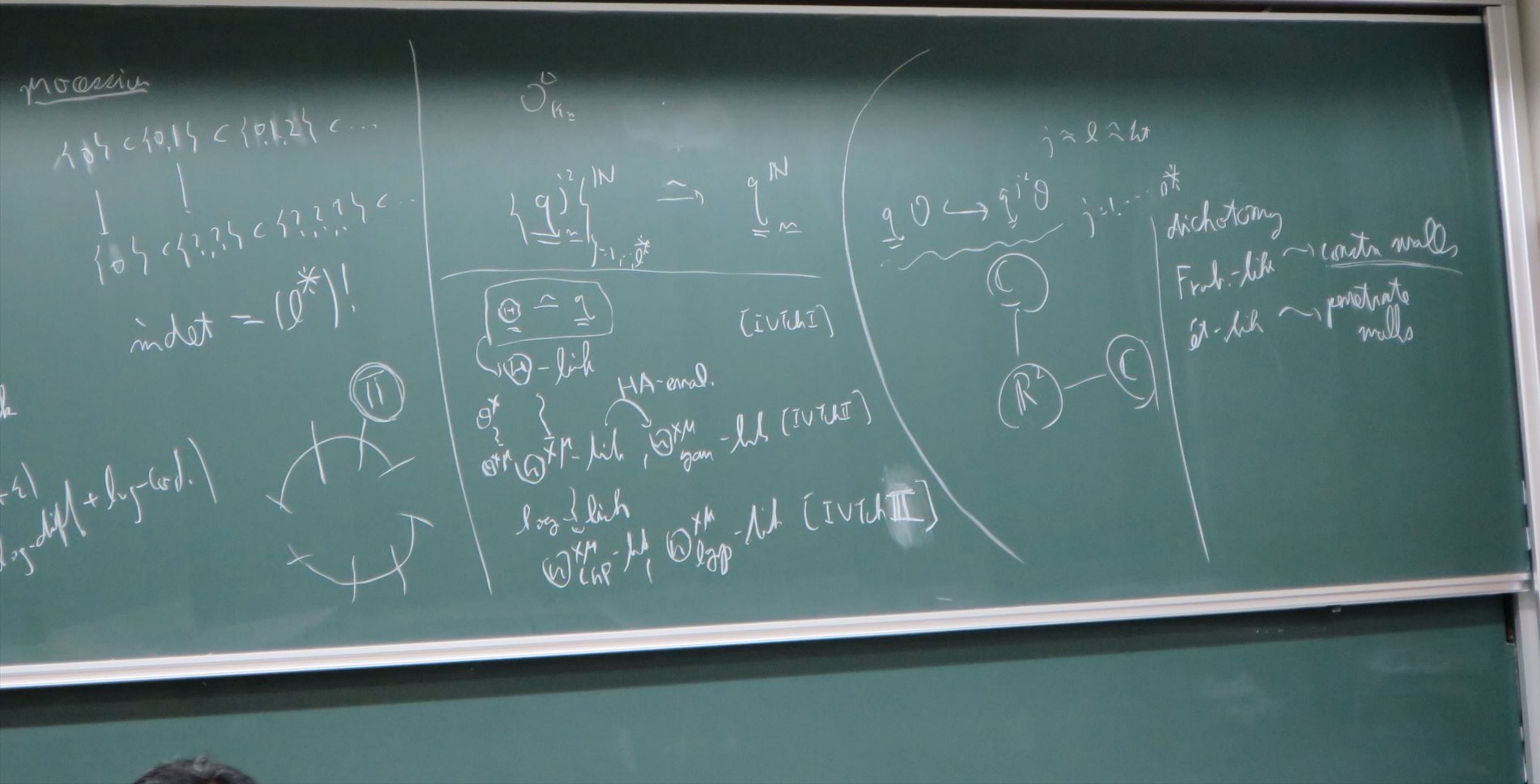
$\Theta^{\times} \rightsquigarrow \Theta^{\times\mu}$

$\Theta^{\times\mu}$-link,

HA-e

log-link

$\Theta^{\times\mu}_{\mathrm{LGP}}$-link,



procession

$\{0\} \subset \{0,1\} \subset \{0,1,2\} \subset \dots$

$\{0\} \subset \{?,?\} \subset \{?,?,?\} \subset \dots$

$\mathrm{indet} = (l^*)!$

k

+ε)

log-diff + log-cond.)

$\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$

$\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}^{\mathbb{N}}_{j=1,\dots,l^*} \xrightarrow{\sim} \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$

$\boxed{\underline{\Theta} \xrightarrow{\sim} \underline{\underline{q}}}$ [IUTch I]

$\hookrightarrow \Theta$-link

$\Theta^{\times}$, $\Theta^{\times\mu}$: $\Theta^{\times\mu}$-link, $\Theta^{\times\mu}_{\mathrm{gau}}$-link [IUTch II]

HA-eval.

log-link

$\Theta^{\times\mu}_{\mathrm{LGP}}$-link, $\Theta^{\times\mu}_{\mathrm{lgp}}$-link [IUTch III]

$j \approx l \approx ht$

$\underline{\underline{q}}\,\mathcal{O} \hookrightarrow \underline{\underline{q}}^{j^2}\mathcal{O}$ $j = 1, \dots, l^*$

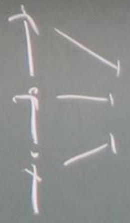
dichotomy

Frob.-like ⇝ constrain walls

ét-like ⇝ penetrate walls

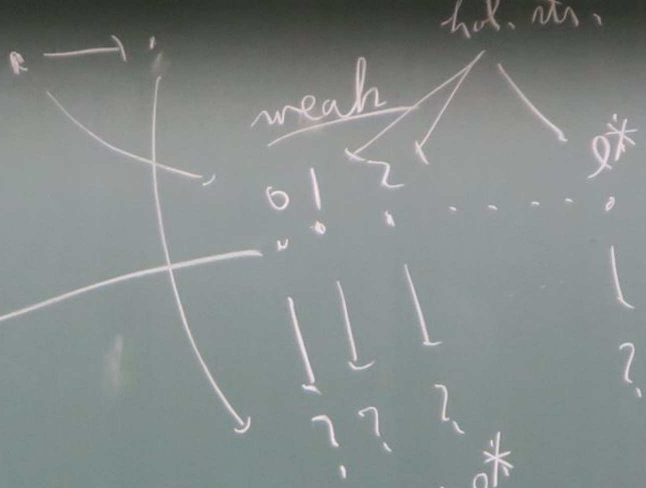
$\to$) $\dagger\Theta^{\times\mu} \xrightarrow{\sim} \ddagger\Theta^{\times\mu}$ up to $\hat{\mathbb{Z}}^{\times}$

$\uparrow$)



upper semi-compact
(common upper bound)

↷) $\dagger\mathcal{G}_{\underline{v}} \xrightarrow{\sim} \ddagger\mathcal{G}_{\underline{v}}$ + Autom of processions



processions

$\{0\} \subset \{0,$

$\{0\} \subset$

ind

$\underline{\log \mathcal{O}^{\times}_{\underline{v}}}$ $\rightsquigarrow$ log-diff.

(Indet)

(log-cond.)

ind det $= (l^*)^{l^*}$

weak

ht $\lesssim \boxed{2}(1+\epsilon)$(log-diff + log-co

mono-

con

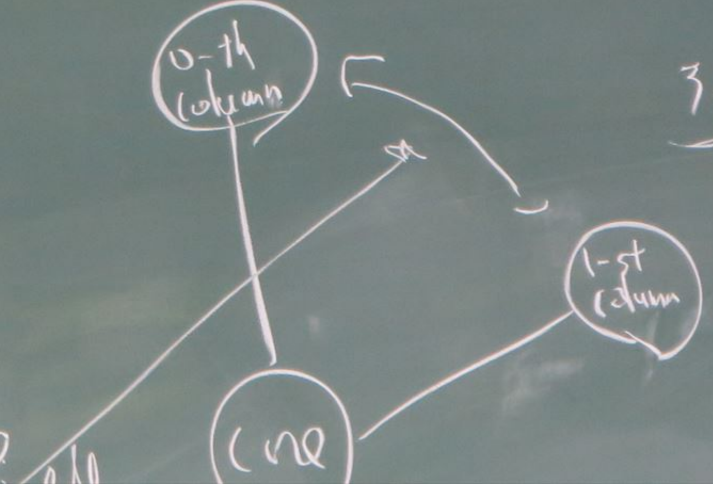
$\{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*} \xrightarrow{\sim} \underline{\underline{q}}$

$\uparrow$ $\uparrow$

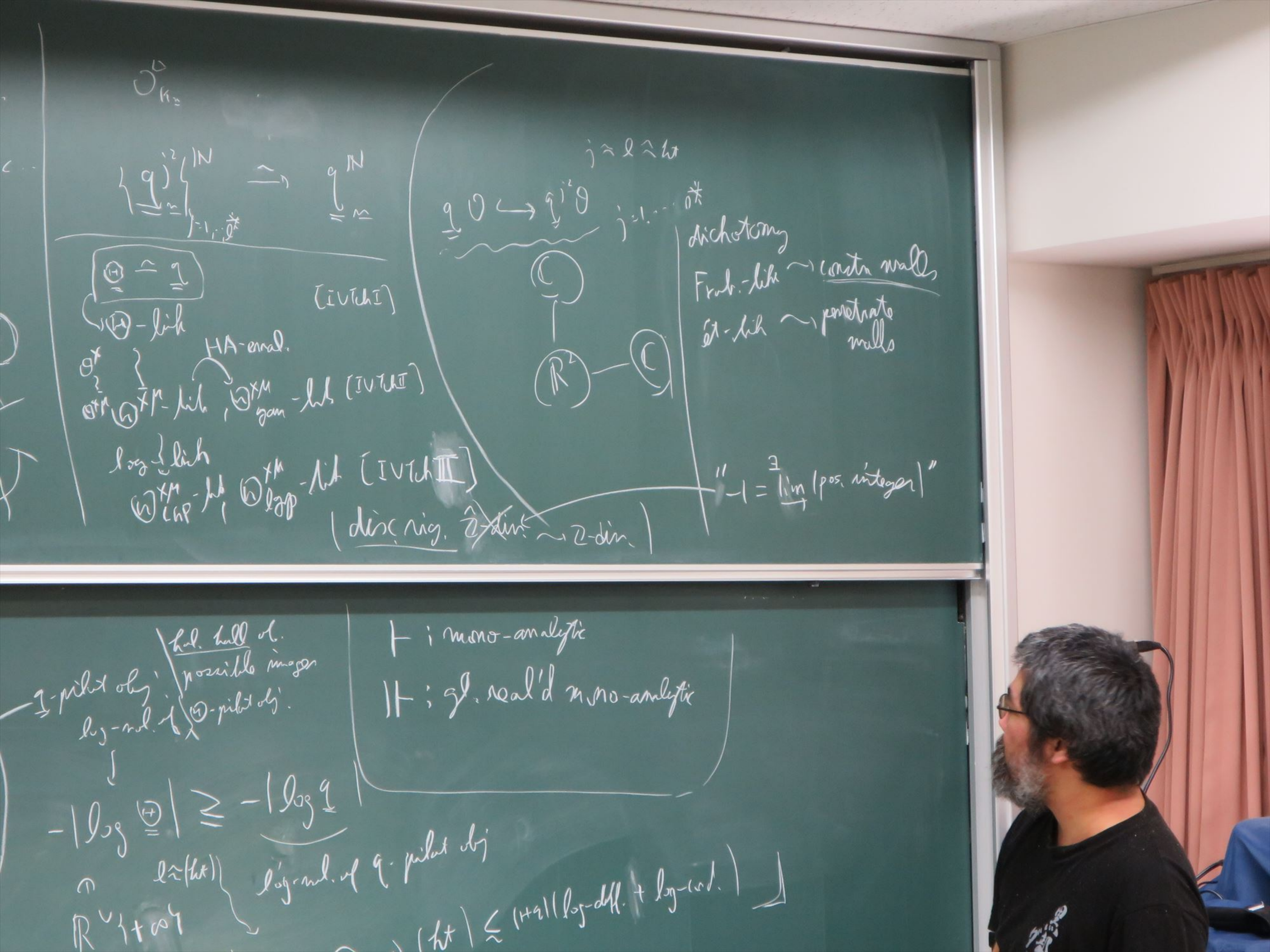
Θ-pilot obj $\xmapsto{\boxtimes\text{-line bdl}}$ q-pilot obj

Frob.

$\{$ log-vol



∃ Indet



$\mathcal{O}^{\triangleright}_{K_{\underline{v}}}$

$\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}^{\mathbb{N}}_{j=1,\dots,l^*} \xrightarrow{\sim} \underline{\underline{q}}^{\mathbb{N}}_{\underline{v}}$

$\boxed{\underline{\underline{\Theta}} \xrightarrow{\sim} \underline{\underline{q}}}$ [IUTch I]

Θ-link

HA-equal.

$\Theta^{\times}$ … $\Theta^{\times\mu}$ $\Theta^{\times\mu}$-link, $\Theta^{\times\mu}_{\mathrm{gau}}$-link [IUTch II]

log-link

$\Theta^{\times\mu}_{\mathrm{LGP}}$-link, $\Theta^{\times\mu}_{\mathrm{lgp}}$-link [IUTch III]

(disc. rig. $\hat{\mathbb{Z}}$-~~dim.~~ ⇝ $\mathbb{Z}$-dim.)

$j \approx l \approx \mathrm{ht}$

$\underline{\underline{q}}\ \theta \hookrightarrow \underline{\underline{q}}^{j^2}\theta \quad j=1,\dots,l^*$

$\mathbb{C}$

$\mathbb{R}^2$ — $\mathbb{C}$

dichotomy

Frob.-like ⇝ constitute walls

ét-like ⇝ penetrate walls

"$-1 = \overset{\exists}{\varinjlim}$ (pos. integer)"

q-pilot obj ; log-vol. of | hol. hull of possible images Θ-pilot obj.

$-|\log \underline{\underline{\Theta}}| \geq -|\log \underline{\underline{q}}|$

$\cap$

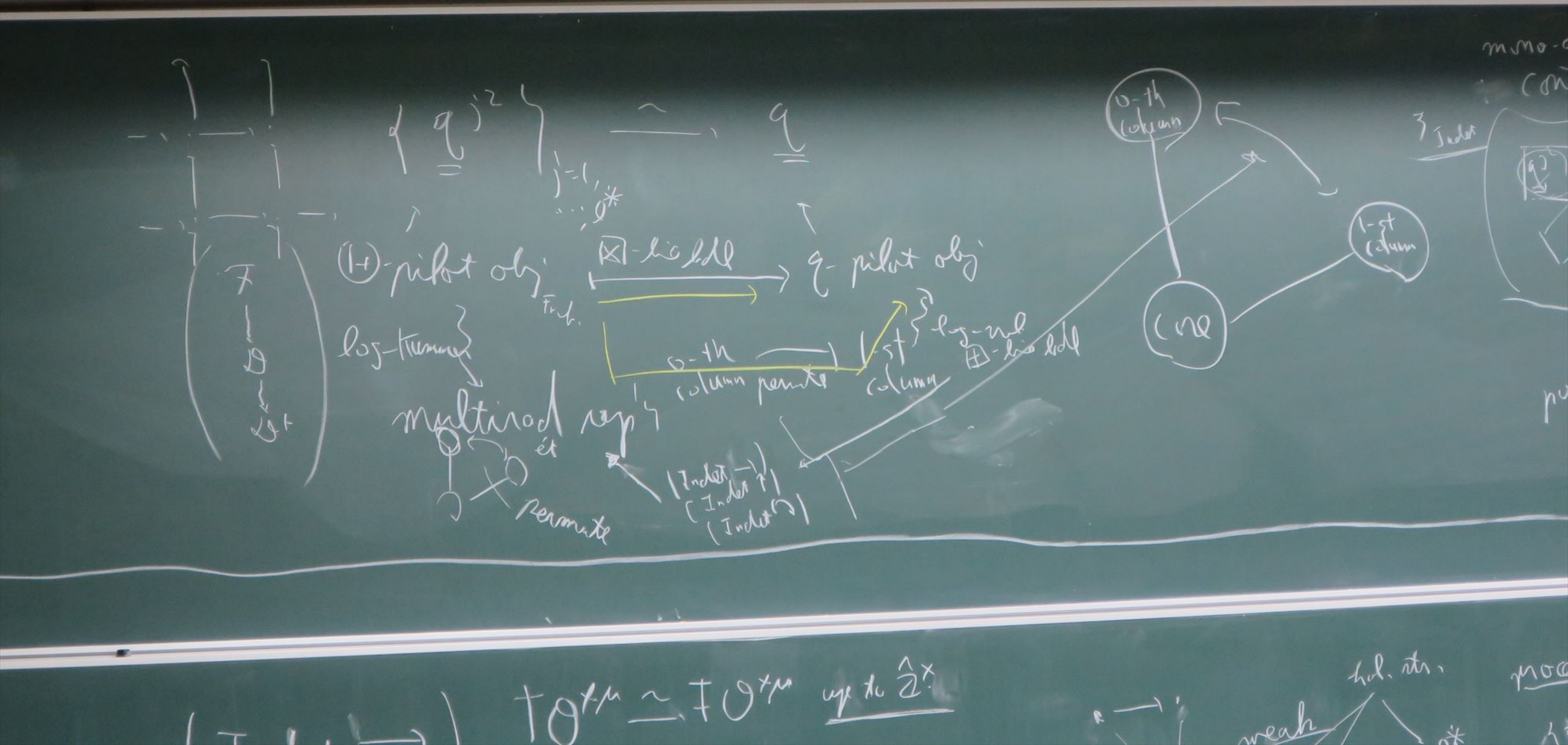
$\mathbb{R} \cup \{+\infty\}$

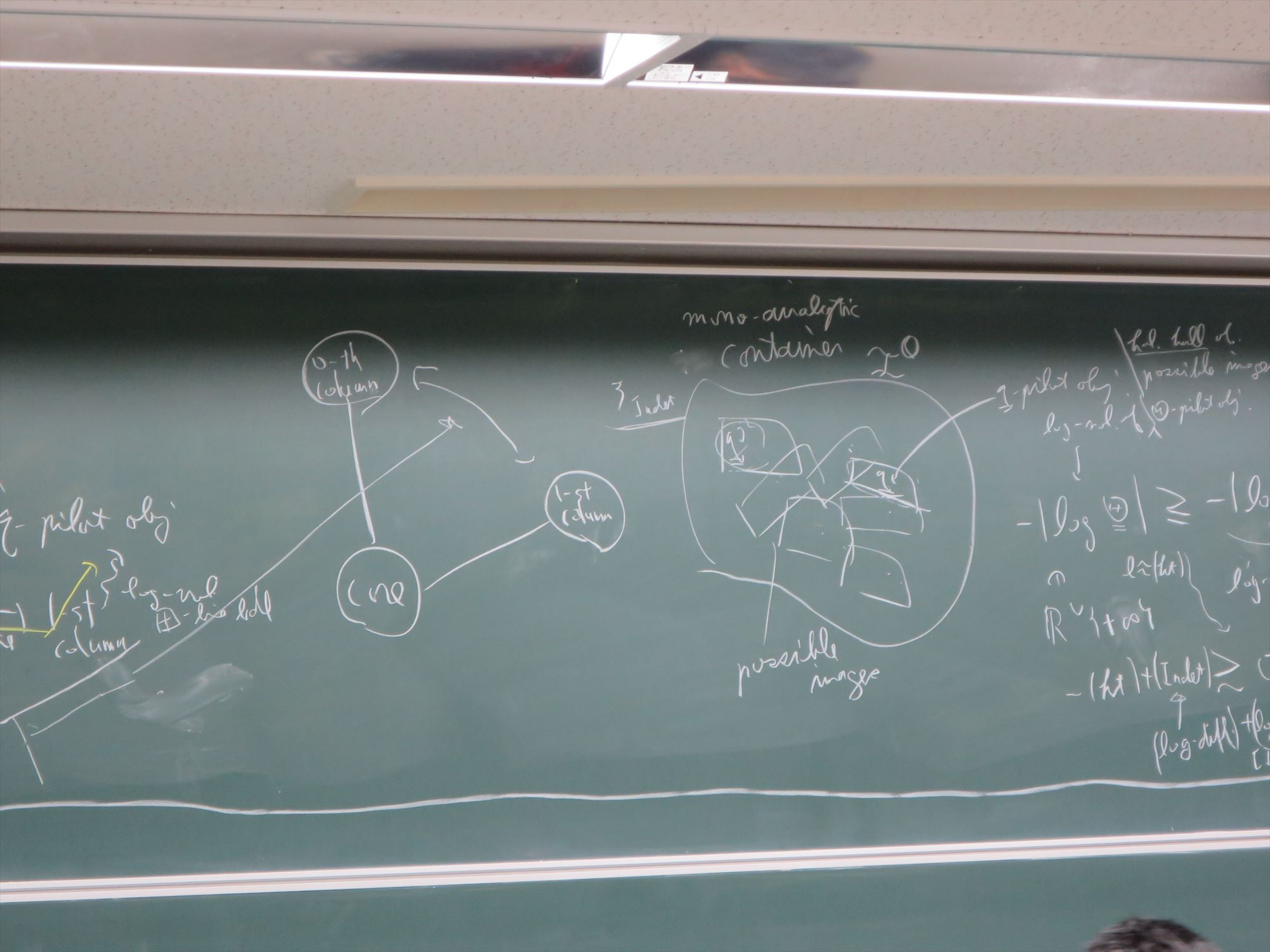
$l \approx (\mathrm{ht})$ log-vol. of q-pilot obj

$(\mathrm{ht}) \lesssim (1+\epsilon)(\text{log-diff.} + \text{log-cond.})$

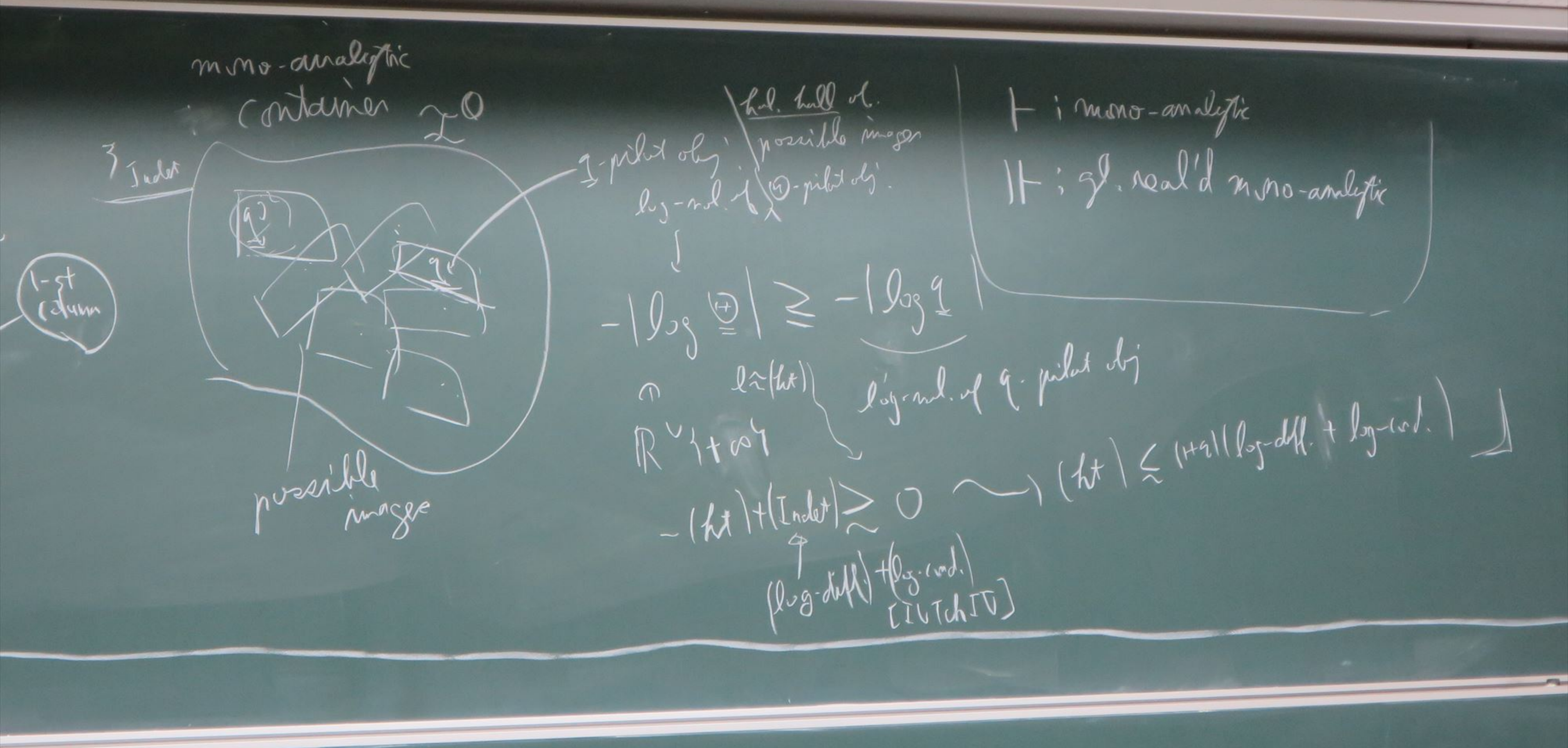
⊢ ; mono-analytic

⊩ ; gl. real'd mono-analytic


$\{\underline{\underline{q}}^{j^2}\}_{j=1,\dots,l^*}$
$\xrightarrow{\sim}$
$\underline{\underline{q}}$
(1)-pilot obj
Frob.
⊠-line bdl
q-pilot obj
log-theta
multirad rep
ét
permute
0-th column
permute
1-st column
log-vol
⊞-line bdl
(Indet →)
(Indet ↑)
(Indet ↻)
0-th column
1-st column
Core
∃ Indet
mono-
hol. str.
weak
$\dagger\mathcal{O}^{\times\mu} \xrightarrow{\sim} \ddagger\mathcal{O}^{\times\mu}$ up to $\hat{\mathbb{Z}}^\times$



0-th column
1-st column
Core
ξ-pilot obj
1-st column
log-vol
⊞-line bdl
mono-analytic container
ℑ Indet
possible images
q-pilot obj
log-vol.
hol. hull of possible images
⊞-pilot obj.
−|log ⊕| ≥ −|lo
ℝ ∪ {+∞}
l≈(ht)
log-
−(ht) + (Indet) ≥
(log-diff) + (lo



mono-analytic container $\mathcal{I}^{\mathbb{Q}}$

$\exists_{\text{Indet}}$

1-st column

q

possible images

q-pilot obj' log-vol. of ⇝ hol. hull of. possible images $\underline{\underline{\Theta}}$-pilot obj'.

$$-|\log(\underline{\underline{\Theta}})| \geq -|\log(\underline{q})|$$

$\cap$

$\mathbb{R} \cup \{+\infty\}$

$\lesssim (\text{ht})$ log-vol. of q-pilot obj

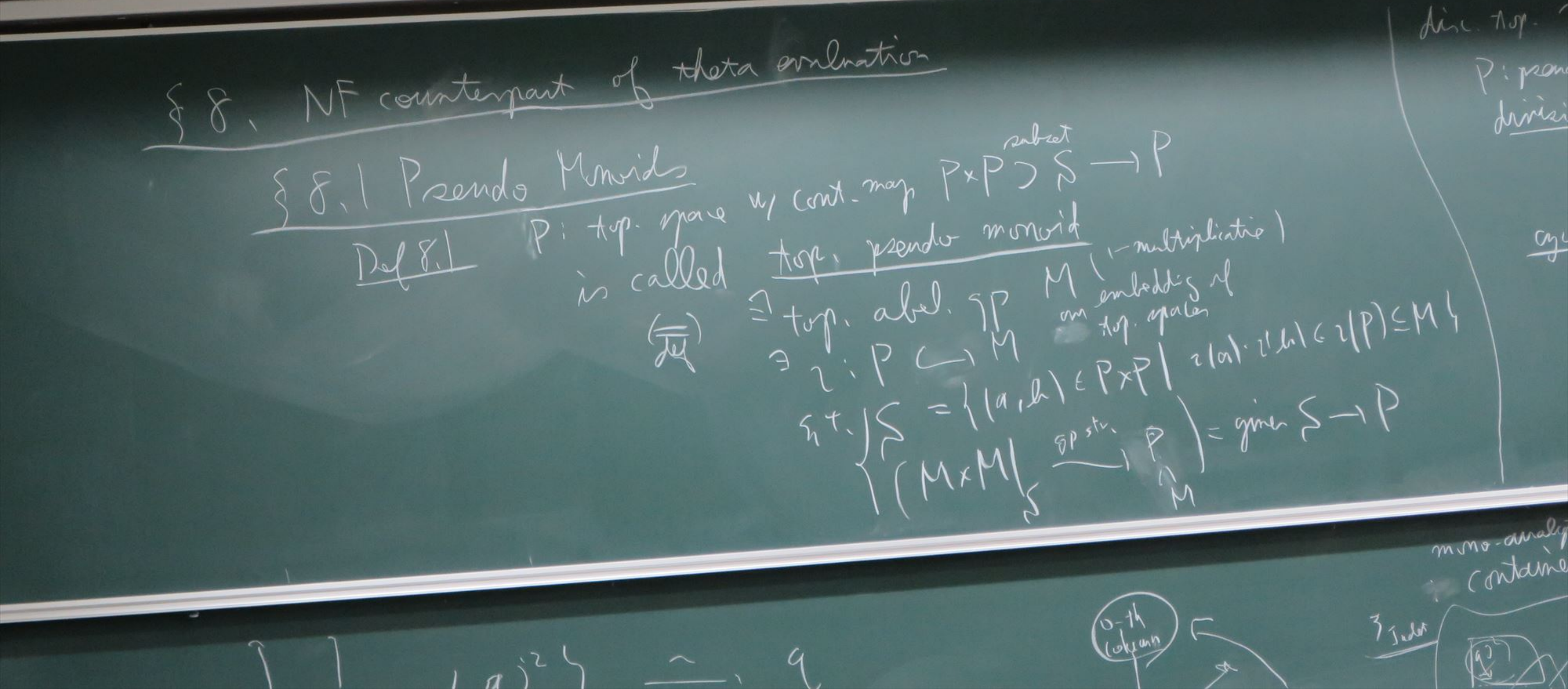
$$-(\text{ht}) + (\text{Indet}) \gtrsim 0 \rightsquigarrow (\text{ht}) \lesssim (1+\epsilon)(\text{log-diff.} + \text{log-cond.})$$

$\uparrow$

(log-diff.) + (log-cond.)
[IUTchIV]

⊢ : mono-analytic

⊩ : gl. real'd mono-analytic



§ 8. NF counterpart of theta evaluation

§ 8.1 Pseudo Monoids

Def 8.1 P: top. space w/ cont. map $P \times P \supset \overset{\text{subset}}{S} \longrightarrow P$

is called top. pseudo monoid ($\leftarrow$ multiplicative)

$\underset{\text{def}}{\Leftrightarrow}$ $\exists$ top. abel. gp M

$\exists\ \iota : P \hookrightarrow M$ an embedding of top. spaces

s.t. $\left\{ \begin{array}{l} S = \{ (a,b) \in P \times P \mid \iota(a) \cdot \iota(b) \in \iota(P) \subseteq M \} \\ \left(M \times M|_S \xrightarrow{\text{gp str.}} \underset{M}{\overset{P}{\cap}} \right) = \text{given } S \to P \end{array} \right.$

disc. top.

P: pseu

divisi

cy

mono-analy

containe

$\exists$ Index

iplicative)

S of

les

$\iota(a)\cdot\iota(b) \in \iota(P) \subseteq M$ }

) = given $S \to P$

disc. top. $\rightsquigarrow$ pseudo-monoid

P : pseudo-monoid

divisible $\overset{\text{def}}{\iff}$ $\exists M, \exists \iota$: as above s.t.

$\forall n \geq 1, \forall a \in M, \exists b \in M, b^n = a$

$\forall n \geq 1, \forall a \in M, a \in \iota(P) \iff a^n \in \iota(P)$

cyclotomic $\overset{\text{def}}{\iff}$ $\exists M, \exists \iota$ as above s.t.

- $\mu_M := M_{tor} (\subset M) \cong \mathbb{Q}/\mathbb{Z}$
- $\mu_M \subseteq \iota(P)$, $\mu_M \cdot \iota(P) \subseteq \iota(P)$

mono-analytic

container $\rightsquigarrow$ Q

hol. hull of

possible images

$\vdash$: mono-analytic

$\Vdash$: gl. real'd

Def 8.2 ([IUTchI, Rem 3.1.7])

$C_{F_{\mathrm{mod}}}$ semi-ell / NF F_{mod} $\quad v$: val of F_{mod}

$C_{F_{\mathrm{mod}}} \leftarrow F_{\mathrm{mod}}$-core

$$L := F_{\mathrm{mod}} \text{ or } (F_{\mathrm{mod}})_v, \quad C_L := C_{F_{\mathrm{mod}}} \times_{F_{\mathrm{mod}}} L \overset{\text{fct field}}{\rightsquigarrow} L_C \subset \underset{\substack{\cup \\ \overline{L}}}{\overline{L_C}}$$

$$\mathring{L} := \begin{cases} F_{\mathrm{mod}} & \text{if } L = F_{\mathrm{mod}} \text{ or } L = (F_{\mathrm{mod}})_v \ v: \text{non-Arch.} \\ (F_{\mathrm{mod}})_v & \text{if } L = (F_{\mathrm{mod}})_v \ v: \text{Arch.} \end{cases}$$

fct field

$\hookrightarrow L_c \subset \overline{L_c}$

$\overline{L}_c \subset$

$|_n$ is non-Arch.

h.

$$\overline{L_c} \supset ⑪ \underset{\text{fin.}}{\supset} L_c$$

⑪ → proper curve

$\cup$

closed pt is called

⑪ $\subset$ (cpt'fication of $C_{F_{m,d}}$)

(1).

a critical point $\overset{\text{def}}{\Longleftrightarrow}$ ⑪ ← 2-tors pt of ell. curve

comes from a

a strictly critical point $\overset{\text{def}}{\Longleftrightarrow}$

- critical
- does not come from cusp

(2).

$\overset{\text{def}}{\Longleftrightarrow}$

ed

→ ⓟ ∈ (cpt'tion of $C_{F_{m,d}}$)

↖ 2-tors pt of ell. curve

es from a

$\overset{def}{\iff}$ { - critical

- does not come from cusp

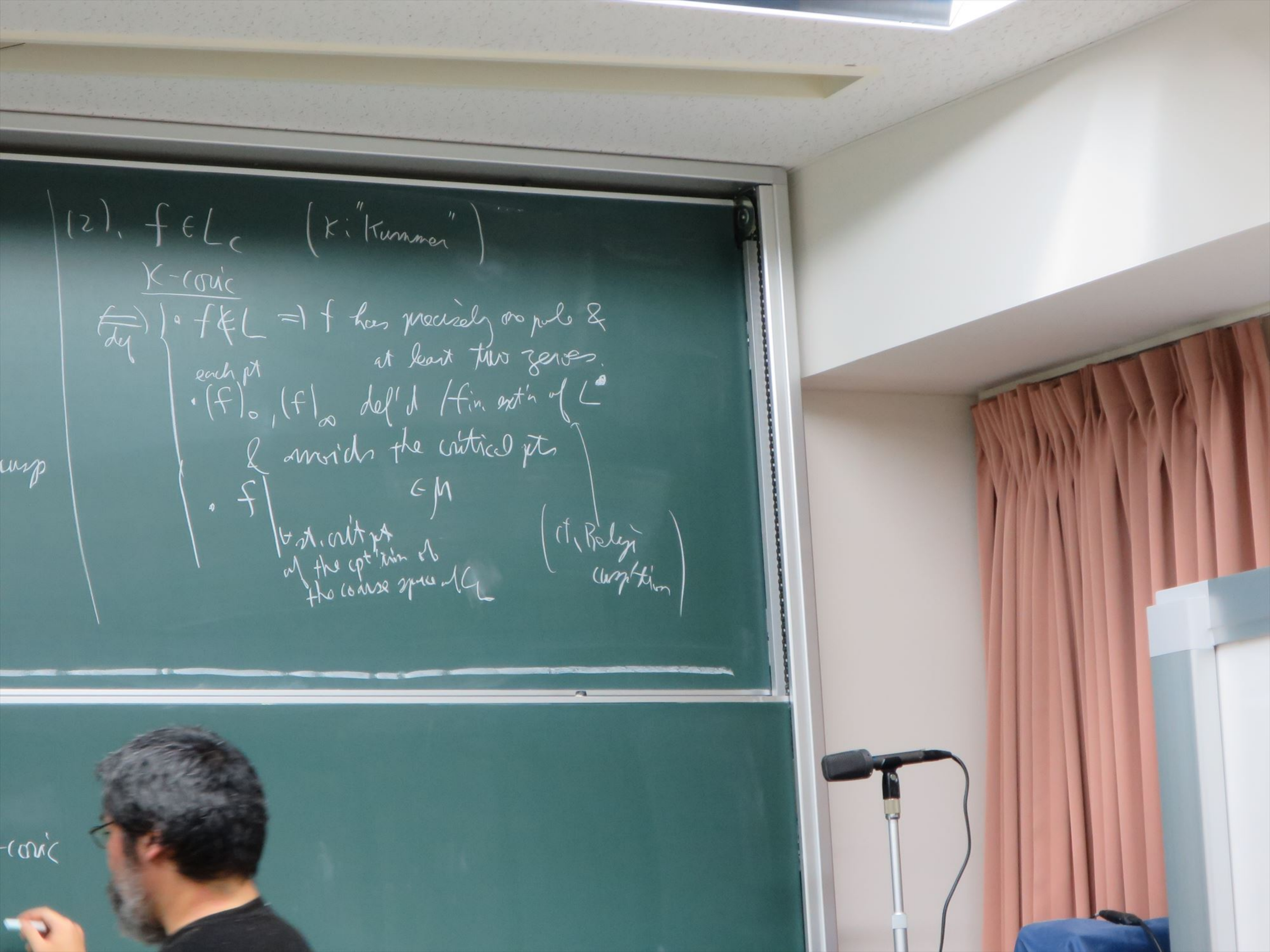
(2). $f \in L_c$ (K: "Kummer")

<u>K-conic</u>

$\overset{def}{\iff}$ { • $f \notin L \Rightarrow f$ has precisely one pole & at least two zeros.

• each pt $(f)_0$, $(f)_\infty$ def'd / fin. ext'n of L & avoids the critical pts $\in M$

• $f \big|_{\forall\ \text{st. crit pt of the cpt'tion of the coarse space of } C_L}$



(2). $f \in L_c$ (κ: "Kummer") ...

κ-conic

$(\frac{\Longleftarrow}{dy})$ • $f \notin L \Rightarrow f$ has precisely one pole & at least two zeros.

each pt

• $(f)_0, (f)_\infty$ def'd / fin. ext'n of L

& avoids the critical pts $\in M$

• f | ∀ pt. crit pt of the cpt'tion of the coarse space of C_L

(cf. Belyi compt'tion)

cusp

-conic

Note $f \notin L \Rightarrow$ never both f, f^{-1} : K-conic

$c \in L$, $f \in L_c$, f, cf : K-conic $\Rightarrow c \in \mu$

$\forall$ elt $\in L$ (resp. $\bar{L}$) appears as a value of
some K-conic rat. fct. on C_L
at some non-critical L-(resp. $\bar{L}$-) valued pt of C_L
(by Hensel's lemma)

(3

$\mathcal{C}_L$

valued

of $\mathcal{C}_L$

(3). $\mathcal{U}_{\overline{L}} := \begin{cases} \overline{L}^{\times} & \text{if } L = F_{\mathrm{mod}} \\ \mathcal{O}_{\overline{L}}^{\times} & \text{if } L = (F_{\mathrm{mod}})_m \end{cases}$

$f \in \overline{L}_c$ $\quad \underline{\infty k\text{-conic}} \overset{\text{def}}{\iff} \exists_{n \geq 1},\ f^n : k\text{-conic}$

$\underline{\infty k^{\times}\text{-conic}} \overset{\text{def}}{\iff} \exists\ c \in \mathcal{U}_{\overline{L}}$ s.t. $cf : \infty k\text{-conic}$

$($k-conic nat. fct$) \longleftarrow$

$\cap$

$($$\infty k$-conic nat. fct$) \longleftarrow$

$\cap$

$($$\infty k^{\times}$-conic nat. fct$)$

Note
- $f \in L_c$: k-conic $\iff$ ∞k-conic
- $f \in \overline{L}_c$: $\infty k^{\times}$-conic in

  ∞k-conic $\iff$ $f|_{\text{the crit. pt of proper curve}} \in M$

  $f \in ⑪ \supset_{\text{fin.}} L_c$

(12). $f \in L_c$ $\quad ($k

$\overline{L}^{\times}$ if $L = F_{\mathrm{mod}}$

$O^{\times}_{\overline{L}}$ if $L = (F_{\mathrm{mod}})_m$

∞K-conic $\overset{\text{def}}{\iff}$ $\exists n \geq 1$, f^n : K-conic

∞KX-conic $\overset{\text{def}}{\iff}$ $\exists c \in U_{\overline{L}}$ s.t. cf : ∞K-conic

: K-conic $\iff$ ∞K-conic

: ∞KX-conic $\Downarrow$

∞K-conic $\iff$ $f|_{\text{str. curve at } \infty \text{ of proper curve}} \in M$

$f \in \bigoplus_{\text{fin.}} \mathbb{Z} \cdot L_C$

$$\begin{tikzcd}
\{K\text{-conic nat. fct}\} & & \text{ps. monoid} \arrow[ll] \\
\{\infty K\text{-conic nat. fct}\} \arrow[u, phantom, "\cap"] & & \text{dim. eql. ps. monoids} \arrow[ll] \arrow[dll] \\
\{\infty KX\text{-conic nat. fct}\} \arrow[u, phantom, "\cap"] & &
\end{tikzcd}$$

§ 8.2 Cycl. Rig via Elem. Number Theory

$K = F(E[\ell])$

$$\begin{tikzcd}
\underline{X}_K \arrow[r] \arrow[d, "\ell/2"'] & C_K \\
\underline{C}_K &
\end{tikzcd}$$

as in §7 $\quad C_K := C_F \times_F K$

$C_F := (E_F \setminus \{0\}) / \{\pm 1\}$

$\quad \ \ 1_{F:NF}$

ⓗ-approach ([IUTch I, Rem 3.1.2])

$\Pi_{\underline{C}_K} \supset \Pi_{\underline{X}_K}$

do not use mono-anab. recon. alg'm in §3 directly to $\Pi_{\underline{C}_K}$

but first recon $\Pi_{\underline{C}_K} \rightsquigarrow \Pi_{\underline{X}_K}$

then, use mono-anab. recon. alg'm in §3

to $\Pi_{\underline{X}_K}$ w/ Gal$(\underline{X}_K / \underline{C}_K)$-action

$\Pi_{\underline{C}_K} \supset \Delta_{\underline{C}_K}$

(decomp. gp of non-zero labelled cusp

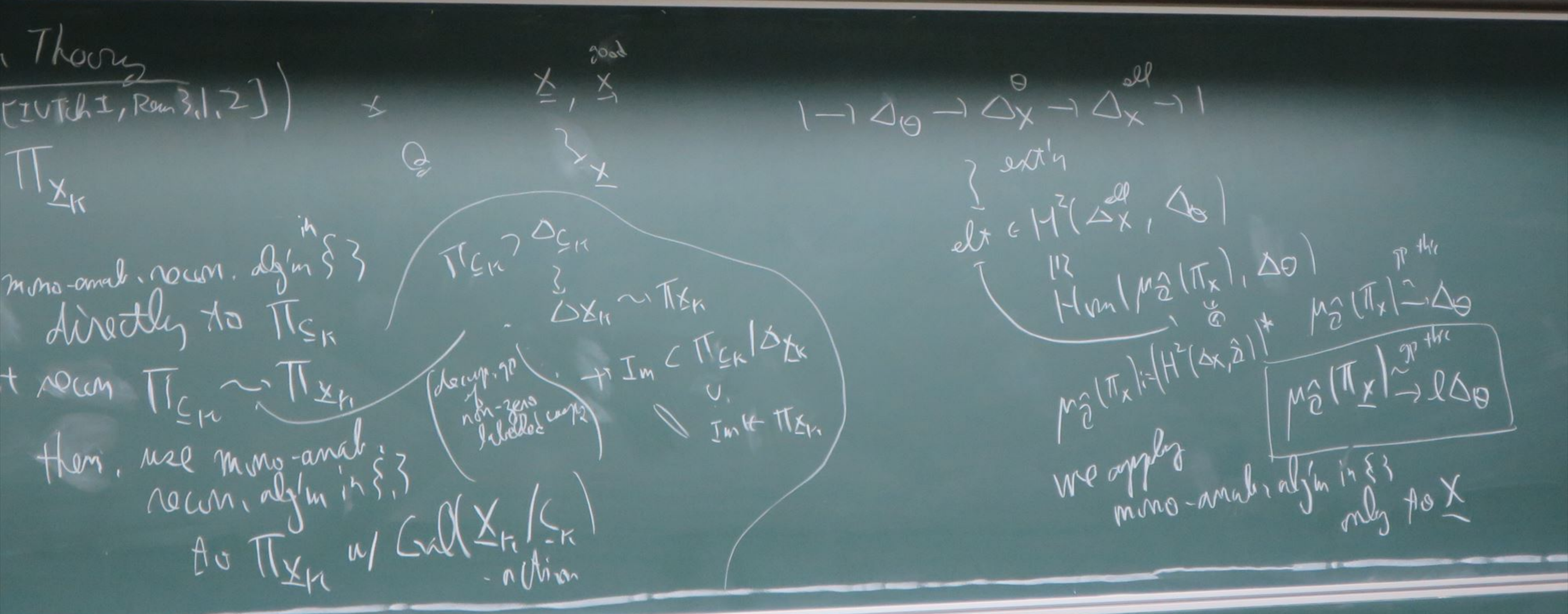
Note $f \notin L \Rightarrow$ never both f, f^{-1}: K-conic

$c \in L$, $f \in L_c$, f, cf : K-conic $\Rightarrow c \in \mu$

$\forall$ (resp. $\overline{L}$) appears as a value of

(3). $\mathcal{U}$

f



Theory

[IUTch I, Rem 3.1.2])

$\Pi_{\underline{X}_K}$

mono-anab. recon. alg'm $\overset{\text{in}}{}$ §3

directly to $\Pi_{\underline{C}_K}$

t recon $\Pi_{\underline{C}_K} \rightsquigarrow \Pi_{\underline{X}_K}$

then, use mono-anab. recon. alg'm in §3

to $\Pi_{\underline{X}_K}$ w/ $\mathrm{Gal}(\underline{X}_K/\underline{C}_K)$-action

$\underline{\underline{X}}$, $\overset{\text{good}}{\underline{\vec{X}}}$

$\rightsquigarrow \underline{X}$

$\Pi_{\underline{C}_K} \supset \Delta_{\underline{C}_K}$

$\wr$

$\Delta_{\underline{X}_K} \rightsquigarrow \Pi_{\underline{X}_K}$

$\rightarrow \mathrm{Im} \subset \Pi_{\underline{C}_K}/\Delta_{\underline{X}_K}$

$\cup$

$\mathrm{Im} \twoheadleftarrow \Pi_{\underline{X}_K}$

(decomp. gp of non-zero labelled cusp)

$$1 \to \Delta_\Theta \to \Delta_X^{\Theta} \to \Delta_X^{\mathrm{ell}} \to 1$$

$\wr$ ext'n

$\mathrm{el}^* \in H^2(\Delta_X^{\mathrm{ell}}, \Delta_\Theta)$

$\wr|$

$\mathrm{Hom}(\mu_{\hat{\mathbb{Z}}}(\Pi_X), \Delta_\Theta)$

$\mu_{\hat{\mathbb{Z}}}(\Pi_X) \xrightarrow[\sim]{\text{gp thic}} \Delta_\Theta$

$\mu_{\hat{\mathbb{Z}}}(\Pi_X) := (H^2(\Delta_X, \hat{\mathbb{Z}}))^*$

$$\boxed{\mu_{\hat{\mathbb{Z}}}(\Pi_{\underline{X}}) \xrightarrow[\sim]{\text{gp thic}} l\Delta_\Theta}$$

we apply mono-anab. alg'm in §3 only to $\underline{X}$

$$\mathcal{D}^{\circledcirc} := \mathbb{B}(\underline{C}_K)^{0}$$

$$\wr\wr$$

$\underline{X}$ $\circledcirc^{\pm}$ $\quad {}^{\dagger}\mathcal{D}^{\circledcirc}$

$\underline{C}$ $\circledcirc$ global

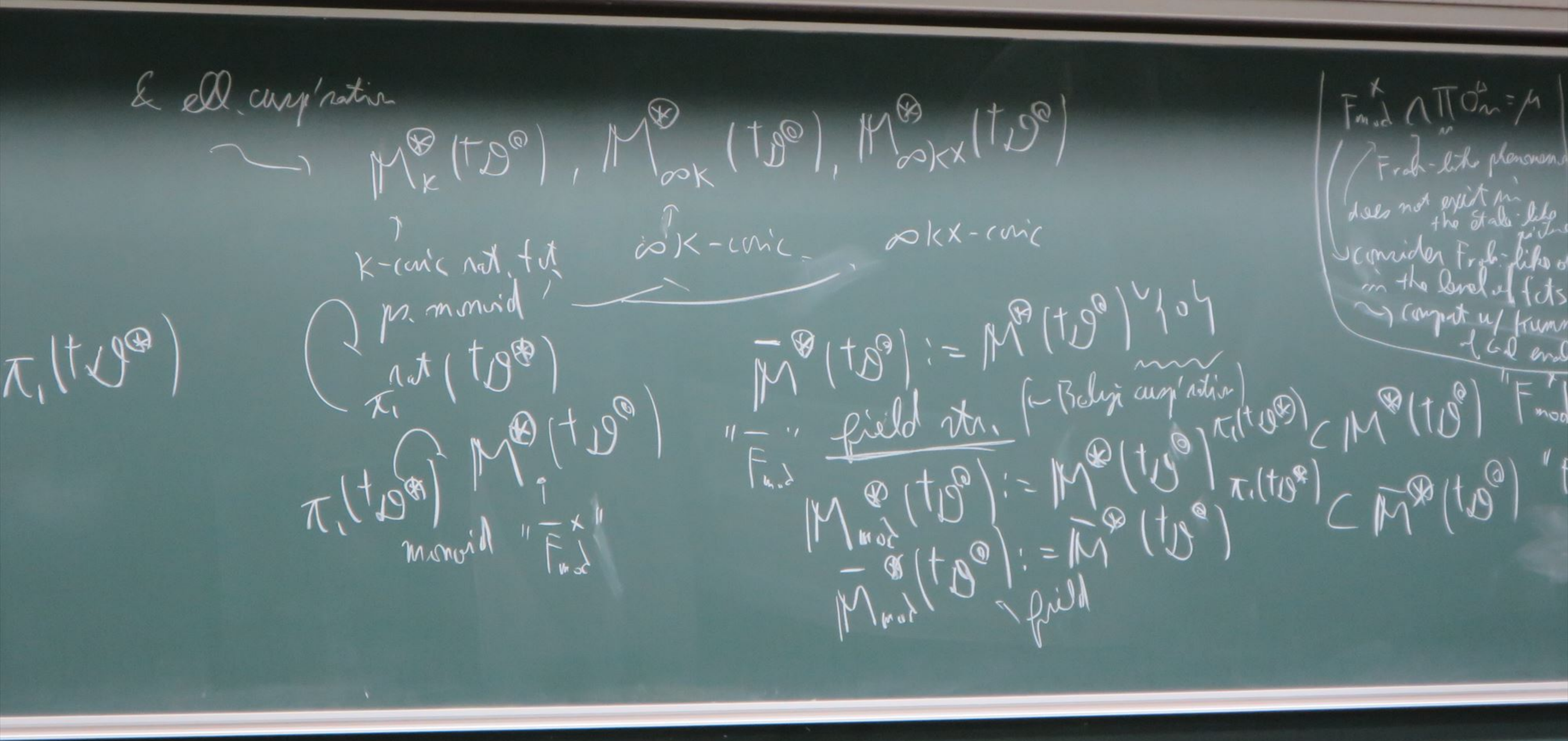
$C_{F_{\mathrm{mod}}}$ $\circledast$ hyperb. core, global

$$\begin{tikzcd}
{}^{\dagger}\mathcal{D}^{\circledcirc} \arrow[r, "\text{recon}"'] & {}^{\dagger}\mathcal{D}^{\circledast} \arrow[dr, bend left] & \\
& \text{corresp. } \mathbb{B}(C_{F_{\mathrm{mod}}}) \arrow[u] & \pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \arrow[r, two heads] & \pi_1({}^{\dagger}\mathcal{D}^{\circledast})
\end{tikzcd}$$

alg.
Gal gp'd
rat. fct

Belyi cusp'tion - π_1^{rat}

& ell. c



& ell. cusp'ation

$\rightsquigarrow \mathbb{M}^{\circledast}_{k}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{\infty k}({}^{\dagger}\mathcal{D}^{\circledcirc}),\ \mathbb{M}^{\circledast}_{\infty k \times}({}^{\dagger}\mathcal{D}^{\circledcirc})$

$\uparrow$ k-conic nat. fct. / pr. monoid — $\uparrow$ ∞k-conic — $\infty k \times$-conic

$\pi_1({}^{\dagger}\mathcal{D}^{\circledast})$

$\pi_1^{\mathrm{nat}}({}^{\dagger}\mathcal{D}^{\circledast})$

$\pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

monoid $\text{"}\overline{F}^{\times}_{\mathrm{mod}}\text{"}$

$\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \cup \{0\}$

$\text{"}\overline{F}_{\mathrm{mod}}\text{"}$ field str. (← Belyi cusp'ation)

$\mathbb{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \mathbb{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ $\text{"}F^{\times}_{\mathrm{mod}}$

$\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ "

field

$F^{\times}_{\mathrm{mod}} \cap \prod_{v} \mathcal{O}^{\triangle}_{v} = \mu$

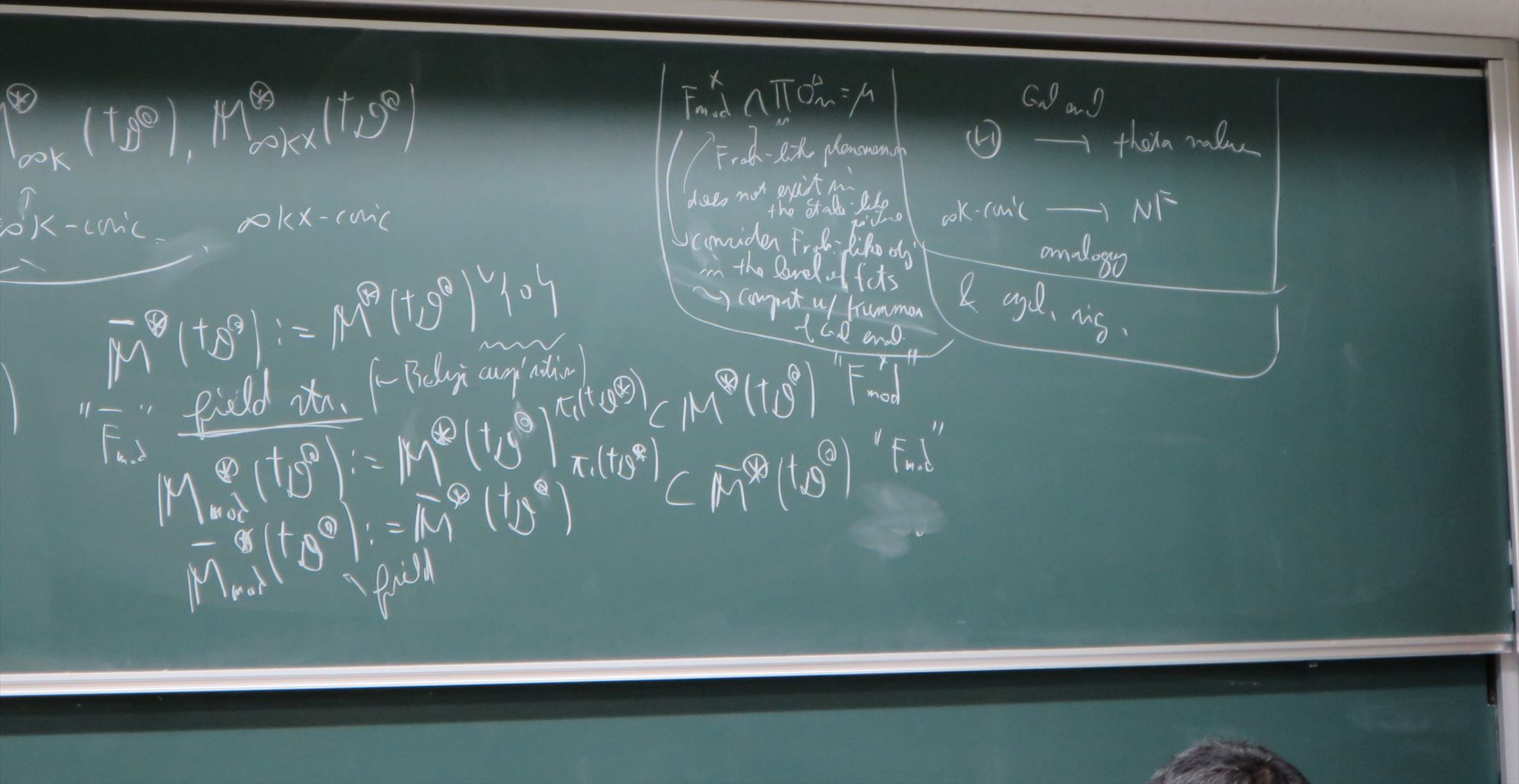
Frob.-like phenomena

does not exist in the étale-like picture

consider Frob.-like ob on the level of fcts

$\rightsquigarrow$ compat w/ Kumm

& [illegible] end



$\mathcal{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}), \mathcal{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc})$

↑

∞κ-coric ∞κ×-coric

$\overline{\mathcal{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathcal{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \cup \{0\}$

"—" F_{mod} field str. (← Belyi cuspidalization)

$\mathcal{M}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \mathcal{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \mathcal{M}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ "$F^{\times}_{\mathrm{mod}}$"

$\overline{\mathcal{M}}^{\circledast}_{\mathrm{mod}}({}^{\dagger}\mathcal{D}^{\circledcirc}) := \overline{\mathcal{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{\pi_1({}^{\dagger}\mathcal{D}^{\circledast})} \subset \overline{\mathcal{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$ "F_{mod}"

↖ field

$F^{\times}_{\mathrm{mod}} \cap \prod_{v} \mathcal{O}^{\triangle}_{v} = \mu$

Frob-like phenomenon does not exist in the étale-like picture

consider Frob-like obj' in the level of fcts

⤳ compat w/ Kummer & Gal eval.

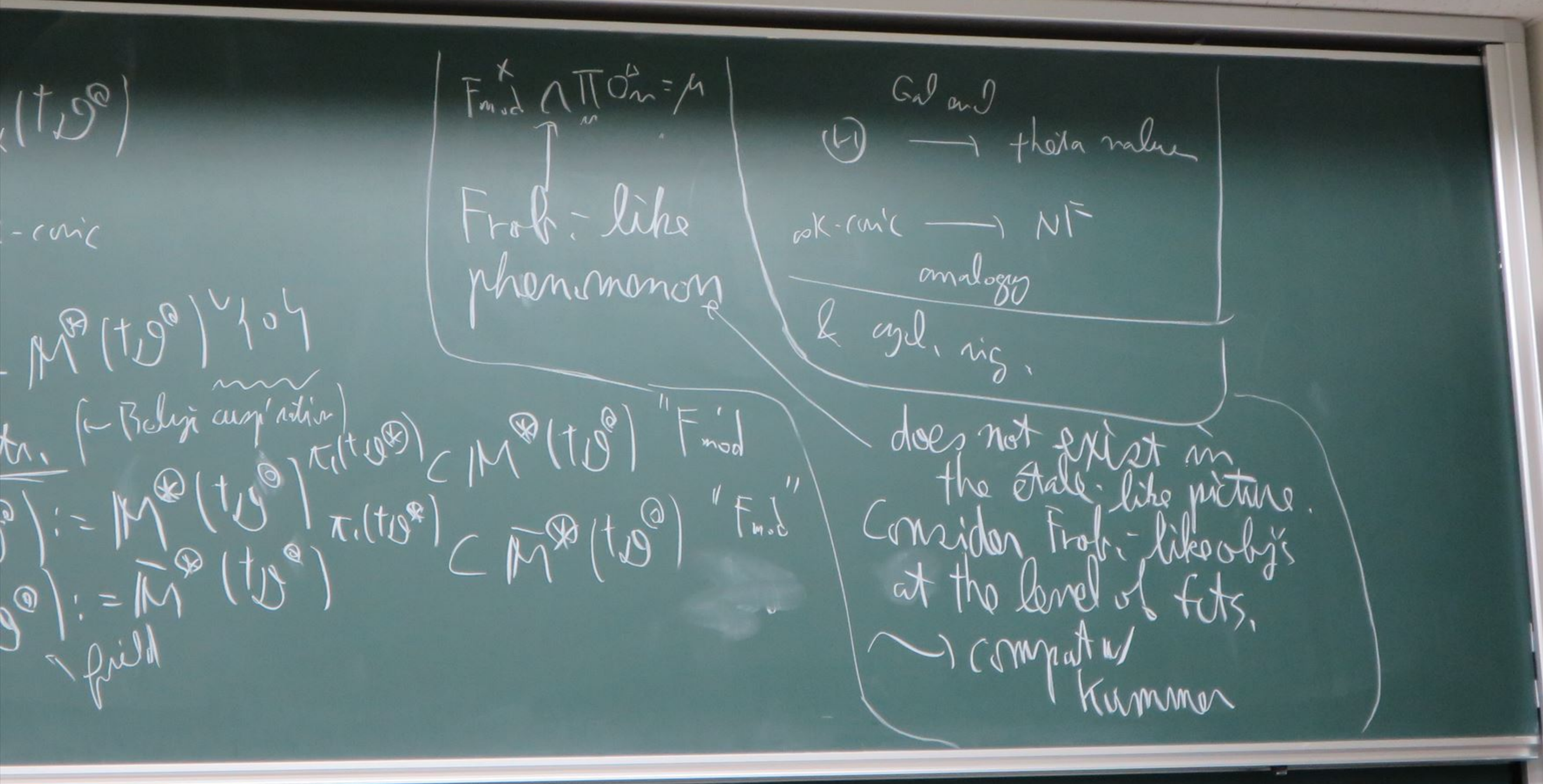
Gal eval

(⊢1) ⟶ theta value

∞κ-coric ⟶ NF

analogy

& cycl. rig.



$(\dagger\mathcal{D}^{\circledcirc})$

-coric

$\mathcal{M}^{\circledast}(\dagger\mathcal{D}^{\circledcirc}) \cup \{0\}$

tr. (← Belyi cuspidalization)

$) := \mathcal{M}^{\circledast}(\dagger\mathcal{D}^{\circledcirc})^{\pi_1(\dagger\mathcal{D}^{\circledast})} \subset \mathcal{M}^{\circledast}(\dagger\mathcal{D}^{\circledcirc})$ "F_{mod}"

$) := \overline{\mathcal{M}}^{\circledast}(\dagger\mathcal{D}^{\circledcirc})^{\pi_1(\dagger\mathcal{D}^{\circledast})} \subset \overline{\mathcal{M}}^{\circledast}(\dagger\mathcal{D}^{\circledcirc})$ "F_{mod}"

field

$$F^{\times}_{mod} \cap \prod_{v} \mathcal{O}^{\triangle}_{v} = \mu$$

$\uparrow$

Frob.-like phenomenon

Gal eval

$(\text{H}) \longrightarrow$ theta value

$\infty\kappa$-coric $\longrightarrow$ NF

analogy

& cycl. rig.

does not exist in the étale-like picture.

Consider Frob.-like obj's at the level of fcts.

$\rightsquigarrow$ compat w/ Kummer

field str. on $\overline{\mathbb{M}}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

ruby

$\xrightarrow[\text{sp th'c}]{} \overline{\mathbb{V}}({}^{\dagger}\mathcal{D}^{\circledcirc})$ the set of val's
("val. on $\overline{F}_{\mathrm{mod}}$")

$\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}) \rightsquigarrow \overline{\Phi}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$: monoid on ${}^{\dagger}\mathcal{D}^{\circledast}$

$\swarrow$ $A \longmapsto$ stack theoret
arith lin
on $\overline{\mathbb{M}}^{\circledast}($

$\rightsquigarrow$ model Fr'd $\overline{\mathcal{F}}^{\circledast}($

ruby/

$A \longmapsto$ stack theoretic arith. line bdl on $\mathbb{M}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})|^{A}$

$\swarrow$

noid on ${}^\dagger\mathcal{D}^{\circledast}$

$\longrightarrow$ model Fr'd $\mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc})$

Frob.-like $\searrow$ cat. equiv. $\swarrow$ étale-like

given $\left\{\begin{array}{l} {}^\dagger\mathcal{F}^{\circledast} \overset{\sim}{=} \mathcal{F}^{\circledast}({}^\dagger\mathcal{D}^{\circledcirc}) \\ {}^\dagger\mathcal{D}^{\circledcirc} \longrightarrow \mathrm{Base}({}^\dagger\mathcal{F}^{\circledast}) \end{array}\right.$

$$\begin{tikzcd}
C_K & {}^\dagger\mathcal{D}^{\circledcirc} \arrow[d] \\
C_{F_{\mathrm{mod}}} & {}^\dagger\mathcal{D}^{\circledast}
\end{tikzcd}$$

$${}^\dagger\mathcal{F}^{\circledcirc} := {}^\dagger\mathcal{F}^{\circledast}|_{{}^\dagger\mathcal{D}^{\circledcirc}}$$

$${}^\dagger\mathcal{F}^{\circledast}_{\mathrm{mod}} := {}^\dagger\mathcal{F}^{\circledast}|_{\text{ter. obj.}}$$

(base = one morph. cat.)

$$A \in \mathrm{Ob}({}^\dagger\mathcal{F}^{\circledast}) \rightsquigarrow \mathcal{O}^{\times}(A^{\mathrm{birat}})$$

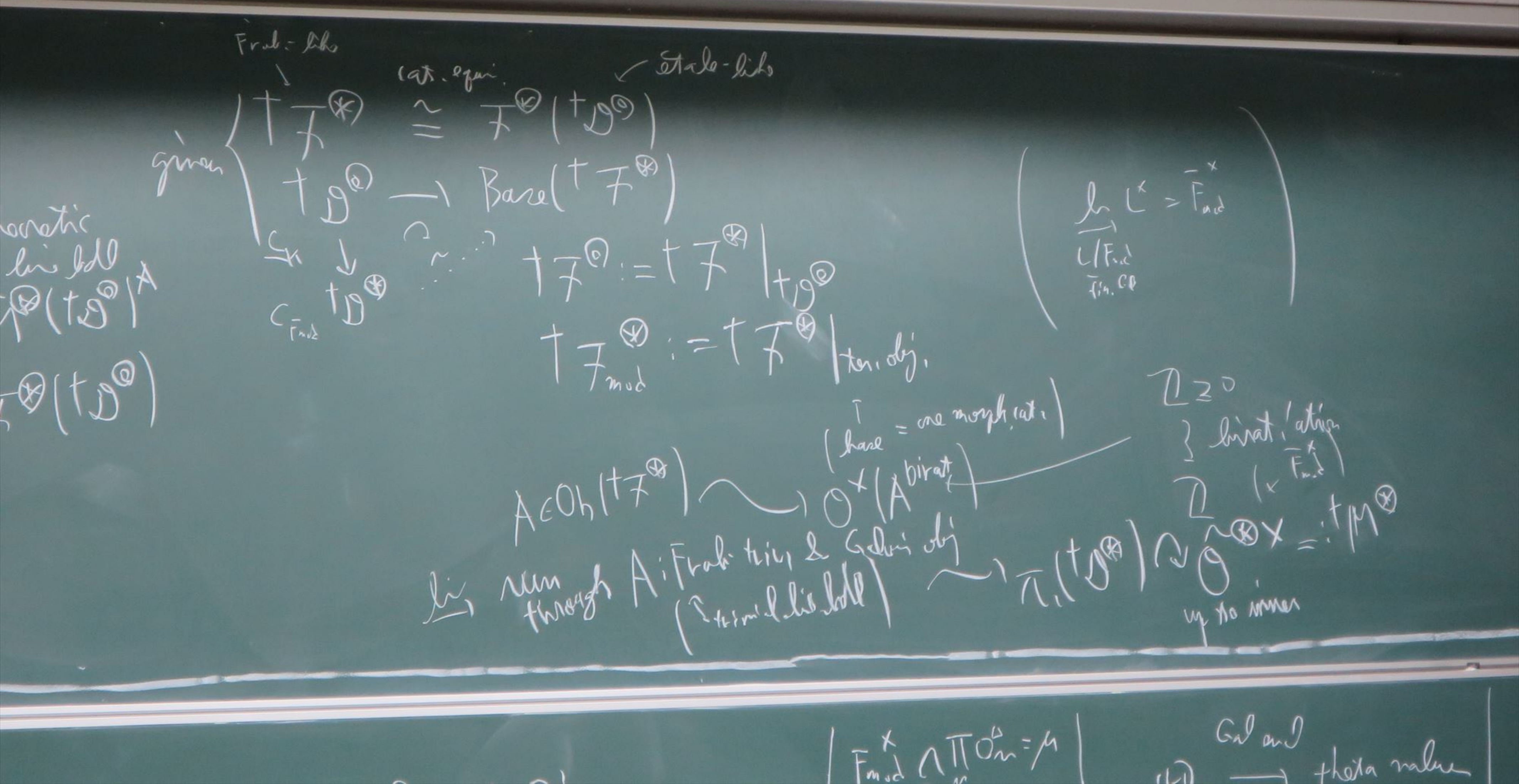
$\varinjlim$, run through A: Frob. triv. & Galois obj (trivial line bdl) $\rightsquigarrow \pi_1({}^\dagger\mathcal{D}^{\circledast}$

$\varinjlim L^{\times}$

L/F_{mod}

fin. CP

$F_{\mathrm{mod}}^{\times} \cap \prod \mathcal{O}^{\triangleright}_{v} = \mu$



Frob.-like

cat. equiv.

étale-like

given
$$\begin{cases} {}^{\dagger}\mathcal{F}^{\circledast} \overset{\sim}{=} \mathcal{F}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc}) \\ {}^{\dagger}\mathcal{D}^{\circledcirc} \longrightarrow \mathrm{Base}({}^{\dagger}\mathcal{F}^{\circledast}) \\ \subseteq_{K} \quad \downarrow \end{cases}$$

$C_{\overline{F}_{mod}} \quad {}^{\dagger}\mathcal{D}^{\circledast}$

$${}^{\dagger}\mathcal{F}^{\circledcirc} := {}^{\dagger}\mathcal{F}^{\circledast}|_{{}^{\dagger}\mathcal{D}^{\circledcirc}}$$

$${}^{\dagger}\mathcal{F}^{\circledast}_{mod} := {}^{\dagger}\mathcal{F}^{\circledast}|_{\text{ter. obj.}}$$

$$\left(\varinjlim_{\substack{L/F_{mod} \\ \text{fin. Gal}}} L^{\times} = \overline{F}^{\times}_{mod} \right)$$

...oretic

... lin. bdl

$\ldots{}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})^{A}$

$\ldots{}^{\circledast}({}^{\dagger}\mathcal{D}^{\circledcirc})$

(base = one morph. cat.)

$$A \in \mathrm{Ob}({}^{\dagger}\mathcal{F}^{\circledast}) \rightsquigarrow \mathcal{O}^{\times}(A^{\mathrm{birat}})$$

$\supseteq 0$

} birat'lization

$(\times \overline{F}^{\times}_{mod})$

$\supseteq$

$$\varinjlim \text{ run through } A:\text{Frob. triv. \& Galois obj} \rightsquigarrow \pi_1({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright \overline{\mathcal{O}}^{\circledast\times} =: {}^{\dagger}\mathcal{M}^{\circledast}$$

(⇔ trivial lin. bdl)

up to inner

$\overline{F}^{\times}_{mod} \cap \prod \mathcal{O}^{\triangle}_{v} = \mu$

Gal eval

theta values

($\Phi_{{}^\dagger\mathcal{F}^\circledast}$: dir. monoid of ${}^\dagger\mathcal{F}^\circledast$)

For $\forall \mathfrak{p} \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^\circledast}(A))$

$$\mathcal{O}^\times(A^{\mathrm{birat}}) \longrightarrow \Phi_{{}^\dagger\mathcal{F}^\circledast}(A)^{\mathrm{gp}}$$

$$\rightsquigarrow \mathcal{O}^\triangleright_\mathfrak{p} \subseteq \mathcal{O}^\times(A^{\mathrm{birat}})$$

$$\mathcal{O}^\times_\mathfrak{p} \subset \mathcal{O}^\triangleright_\mathfrak{p}$$

$$\mathcal{O}^{\times\mathrm{tor}}$$

$\underbrace{A : \text{Frob-triv}, \text{Ga}}$

run through

$\rightsquigarrow$

$\mathrm{Aut}_{{}^\dagger\mathcal{F}^\circledast}($

$A_0 \in \mathrm{Ob}($

lying

$\forall \mathfrak{p}_0 \in$

$\prod$

$\mathcal{F}^{\circledast})$

$A))$

$\circledast(A)^{gp}$

$\mathcal{O}^{\times}(A^{\text{birat}})$

$\mathcal{O}^{\times\text{tor}}$.

A: Frob-trivial, Gal. obj.

run through

$\mathcal{M}$,

$\mathrm{Aut}_{{}^\dagger\mathcal{F}^{\circledast}}(A) \curvearrowright \mathcal{O}^{\times}(A^{\text{birat}})$

permutes $\mathcal{O}^{\triangle}_{\mathfrak{p}}$

$A_0 \in \mathrm{Ob}({}^\dagger\mathcal{F}^{\circledast})$

lying over term. obj.

$\forall\, \mathfrak{p}_0 \in \mathrm{Prime}(\Phi_{{}^\dagger\mathcal{F}^{\circledast}}(A_0))$

$\Pi_{\mathfrak{p}_0} \subseteq {}^\dagger\Pi$ closed sub

well-def up to conj.

decomp. Π fix the submonoid $\mathcal{O}^{\triangleright}_{\mathfrak{p}}$

↑ system of $\mathfrak{p}$'s over $\mathfrak{p}_0$

$\pi_1({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathcal{M}^{\circledast}$

$\pi_1^{\text{rat}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathcal{M}$

$\infty\kappa$-c

($\exists!\ \infty\kappa$-(

We alwa

equip

$\pi_1({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathcal{M}^{\circledast}$

$\pi_1^{\text{rat}}({}^\dagger\mathcal{D}^{\circledast}) \curvearrowright {}^\dagger\mathcal{M}^{\circledast}_{\infty\kappa},\ {}^\dagger\mathcal{M}^{\circledast}_{\infty\kappa\times}$

$\underline{\underline{\infty\kappa\text{-coric}}\ \text{str.}}$ $\underline{\underline{\infty\kappa\times\text{-coric}}\ \text{str.}}$ $\underline{\text{on } {}^\dagger\mathcal{F}^{\circledast}}$

($\exists!$ $\infty\kappa$-($\infty\kappa\times$-) coric str. on ${}^\dagger\mathcal{F}^{\circledast}$

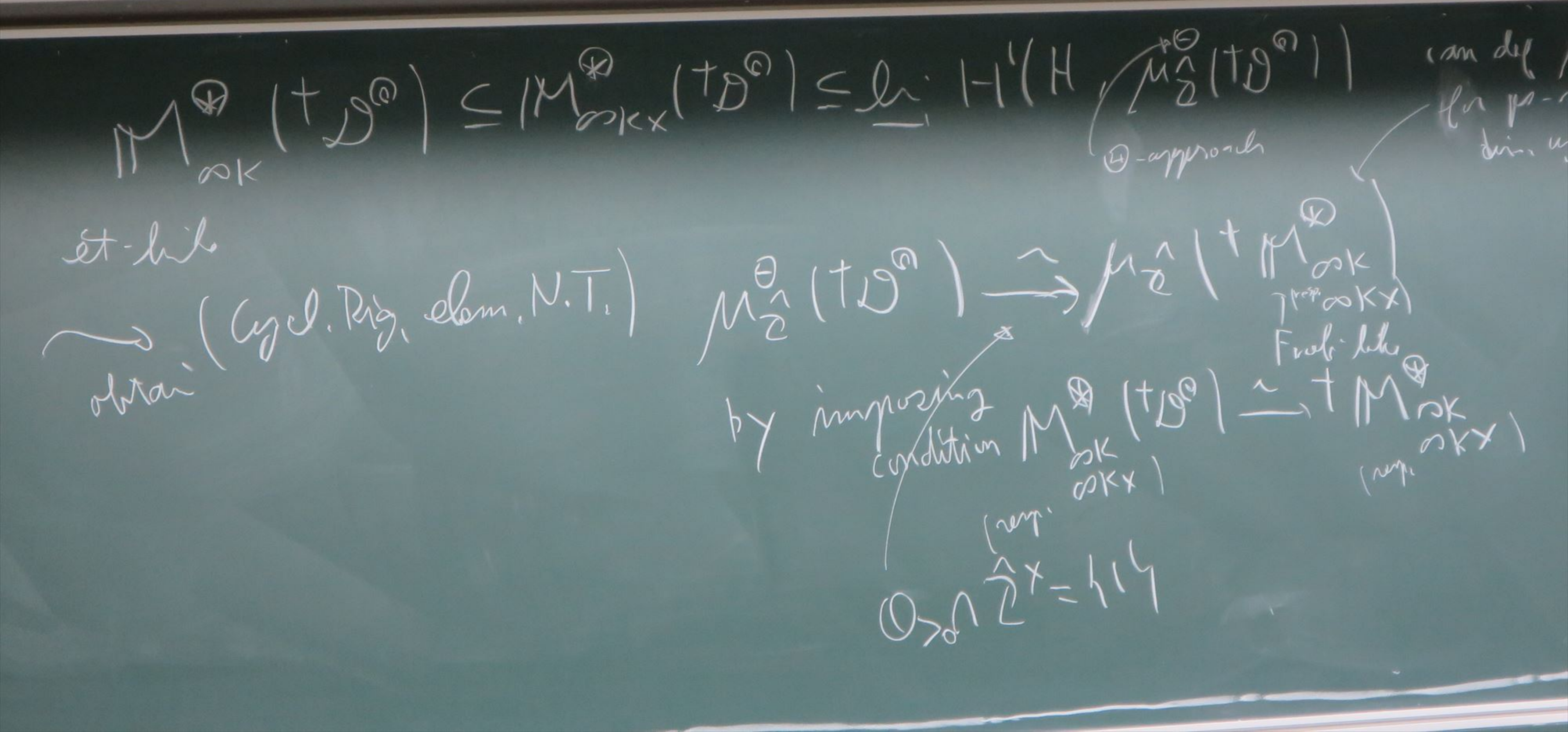
We always consider ${}^\dagger\mathcal{F}^{\circledast}$ as being equipped w/ this unique $\infty\kappa$-($\infty\kappa\times$-)-coric str.)

$${}^\dagger\mathcal{M}^{\circledast}_{\infty\kappa}$$
$$\cup$$
$${}^\dagger\mathcal{M}^{\circledast}_{\kappa} := \left({}^\dagger\mathcal{M}^{\circledast}_{\infty\kappa}\right)^{\pi_1^{\text{rat}}({}^\dagger\mathcal{D}^{\circledast})}$$

$\wp$

stem of $\wp$'s

over $\wp_0$



$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \varinjlim H^1(H, \mu^{\dagger\Theta}_{\hat{\mathbb{Z}}}({}^{\dagger}\mathcal{D}^{\circledcirc}))$$

Θ-approach

can def

for pro-

dim. u

et-like

⇝ obtain (Cycl. Rig. elem. N.T.)

$$\mu^{\Theta}_{\hat{\mathbb{Z}}}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \text{ (resp. } \infty\kappa\times))$$

Frob-like

by imposing condition

$$\mathbb{M}^{\circledast}_{\infty\kappa}({}^{\dagger}\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa} \text{ (resp. } \infty\kappa\times)$$

(resp. $\infty\kappa\times$)

$$\mathbb{Q}_{>0} \cap \hat{\mathbb{Z}}^{\times} = \{1\}$$

$(^{\dagger}\mathcal{D}^{\circledcirc}))$

can def $\mu_{\widehat{\mathbb{Z}}}(-) = \varprojlim_{n} (\ )[n]$
for pv-monoid
div. ugl.

...onch

$^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$
(resp. $\infty\kappa\times$)
Frob-like

$) \xrightarrow{\sim} {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}$
(resp. $\infty\kappa\times$)

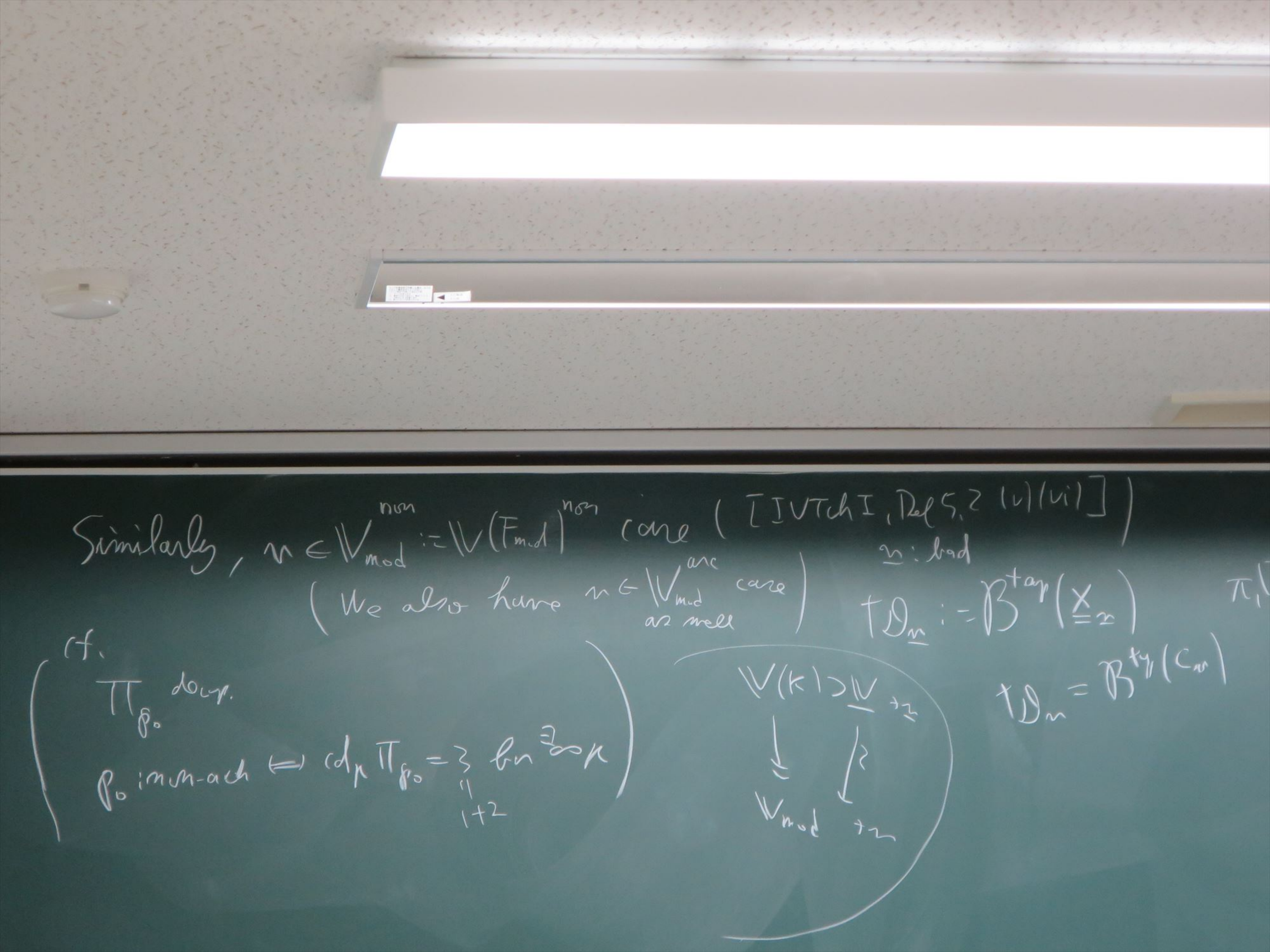
$\infty\kappa\times$-coric str.
considering Kummer classes
restricting to decomp. gp of str. crit pts

$\infty\kappa$-coric str.
consider restricting Kummer classes to decomp. gp of non-crit $\underline{\mathbb{F}_{\mathrm{mod}}\text{-valued}}$ pts
Belyi-cusp'tion

(cf. §3)

field str. on $^{\dagger}\overline{\mathbb{M}}^{\circledast}_{\mathrm{mod}}$

Frob-like	constr. walls
ét-like	penetrate walls
Kummer	



Similarly, $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{non}} := \mathbb{V}(F_{\mathrm{mod}})^{\mathrm{non}}$ case ([IUTchI, Def 5.2 (v)(vi)])

(We also have $v \in \mathbb{V}_{\mathrm{mod}}^{\mathrm{arc}}$ case as well)

$\underline{v}$: bad

$^{\dagger}\mathcal{D}_{\underline{v}} := \mathcal{B}^{\mathrm{tmp}}(\underline{\underline{X}}_{\underline{v}})$

$^{\dagger}\mathcal{D}_{v} = \mathcal{B}^{\mathrm{tp}}(C_{v})$

$\pi_1($

cf. (

$\Pi_{\wp_0}$ decomp.

$\wp_0$: non-arch $\iff$ $\mathrm{cd}_p \Pi_{\wp_0} = \underset{1+2}{\underset{\shortparallel}{3}}$ for $^{\exists}\infty\, p$)

$$\begin{tikzcd}
\mathbb{V}(K) \arrow[d, two heads] \arrow[r, phantom, "\supset"] & \underline{\mathbb{V}} \arrow[dl, "\wr"] \\
\mathbb{V}_{\mathrm{mod}} &
\end{tikzcd}$$

5.2 (v)(vi)])

bad

$:= \mathcal{B}^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})$

$^{\dagger}\mathcal{D}_{v} = \mathcal{B}^{\mathrm{temp}}(C_{v})$

$\pi_1(^{\dagger}\mathcal{D}_{\underline{v}}) \rightsquigarrow \pi_1^{\mathrm{rat}}(^{\dagger}\mathcal{D}_{\underline{v}}) \twoheadrightarrow \pi_1(^{\dagger}\mathcal{D}_{v})$

$\mathcal{M}_{v}(^{\dagger}\mathcal{D}_{\underline{v}}),\ \mathcal{M}_{\kappa v}(^{\dagger}\mathcal{D}_{\underline{v}}),\ \mathcal{M}_{\infty\kappa v}(^{\dagger}\mathcal{D}_{\underline{v}}),\ \mathcal{M}_{\infty\kappa\times v}(^{\dagger}\mathcal{D}_{\underline{v}})$

"$\mathcal{O}^{\triangleright}_{\bar{K}_{\underline{v}}}$" $\curvearrowleft \pi_1(^{\dagger}\mathcal{D}_{v})$

$\curvearrowleft \pi_1^{\mathrm{rat}}(^{\dagger}\mathcal{D}_{v})$

$\cap\ \varinjlim_{H} H^1(H, \mu^{\Theta}_{\hat{\mathbb{Z}}}(^{\dagger}\mathcal{D}_{\underline{v}}$

by im

Tautologically, $\rightsquigarrow\ ^{\dagger}\mathcal{M}_{\kappa v} := (^{\dagger}\mathcal{M}_{\infty\kappa v})^{\pi_1^{\mathrm{rat}}(^{\dagger}\mathcal{D}_{v})}$

$\exists!$ $\infty\kappa$-coric s.th. (Lgcl, Rig / elem. M.T.)

$\infty\kappa\times$-coric

$_{\times}(^{\dagger}\mathcal{D}^{\circledcirc}) \subseteq \varinjlim H^1(H, \mu^{\Theta}_{\hat{\mathbb{Z}}}(^{\dagger}\mathcal{D}^{\circledcirc}))$

can def $\mu_{\hat{\mathbb{Z}}}(-) = \varinjlim_{n} (\ )[n]$

for pf-monoid

tri. ind.

$) \twoheadrightarrow \pi_1({}^\dagger\mathcal{D}_{\underline{v}})$

$, \mathbb{M}_{\kappa\underline{v}}({}^\dagger\mathcal{D}_{\underline{v}}), \mathbb{M}_{\infty\kappa\underline{v}}({}^\dagger\mathcal{D}_{\underline{v}}), \mathbb{M}_{\infty\kappa\times\underline{v}}({}^\dagger\mathcal{D}_{\underline{v}})$

$\cap$

$\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}_{\underline{v}})$

$\mathcal{D}_{\underline{v}})$

${}^\dagger\mathbb{M}_{\kappa\underline{v}} := ({}^\dagger\mathbb{M}_{\infty\kappa\underline{v}})^{\pi_1^{\mathrm{rat}}({}^\dagger\mathcal{D}_{\underline{v}})}$

$\exists!$ $\infty\kappa$-coric str.
$\infty\kappa\times$-coric

(cycl. Rig.
elem. MT.)

Θ-approach
$\downarrow$

$\cap$

$$\varinjlim_H H^1(H, \mu^{\Theta}_{\hat{\mathbb{Z}}}({}^\dagger\mathcal{D}_{\underline{v}}))$$

$$\mu^{\Theta}_{\hat{\mathbb{Z}}}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}({}^\dagger\mathbb{M}_{\substack{\infty\kappa\underline{v}\\ \infty\kappa\times\underline{v}}})$$

by imposing compat. $\mathbb{M}_{\substack{\infty\kappa\underline{v}\\ \infty\kappa\times\underline{v}}}({}^\dagger\mathcal{D}_{\underline{v}}) \to {}^\dagger\mathbb{M}_{\substack{\infty\kappa\underline{v}\\ \infty\kappa\times\underline{v}}}$

$\infty\kappa\times$-cori str.
} str. const
$\infty\kappa$-cori str.
} Frob.-natural non-const. part
build str. on ${}^\dagger\mathbb{M}_{\underline{v}}^{\mathrm{gp}} \cup \{0\}$
$\|$
${}^\dagger\mathbb{M}_{\underline{v}}^{\mathrm{gp}}$

can def. $\mu_{\hat{\mathbb{Z}}}(-) = \varinjlim_n (\ )[n]$
for pf-monoid
divis. cycl.

Frob.-like	const. walls
ét-like	penetrate walls
Kummer	

$\times$-coric str. classes